1.2 MECHANICS OF LAMINATED COMPOSITES.

Two types of mathematical models [1] are normally utilized to predict the stress-strain response of a constituent lamina of a laminated composite — micromodels and macromodels. The micromechanics approach to the problem models the individual lamina as a periodic, or possibly a random, array of filaments in a matrix. The average stress-strain response of the lamina is a function of the elastic constants of the fiber and the matrix and their respective geometries. The macromechanics approach ignores the fiber-matrix behavior and models the individual lamina as a thin homogeneous orthotropic medium (sheet) under a state of plane stress.

1.2.1 MICROMECHANICS.

The strength and structural behavior of fibrous composites are directly related to the elastic properties of the fiber and matrix, as well as the micro-geometry of the laminate [1]. The field of micromechanics encompasses the study of the internal stress distribution in the fiber and matrix as a result of external loading. The objective of any micromechanics effort is to predict the intrinsic macroscopic (average) material properties of a laminate from the material and geometric properties of the constituents and perhaps provide the basis for understanding failure modes and establishing failure criteria from the predicted stress states.

With the present state-of-the-art, a micromechanics approach to composites for aerospace application does not give the designer a design tool which can be utilized to design aerospace structures. It is feasible that at some time in the future, the aerospace designer will begin his creation at the fiber and matrix level and will use the analytical tools of micromechanics to determine his design parameters. However, at present the advanced composites come to the fabrication stage in a preimpregnated tape form with the filament spacing and other parameters established. Of course, some of the predictions resulting from micromechanic analyses were utilized in establishing a good filament spacing to insure good transverse and shear properties in the final laminate; however, this represents a limited application of micromechanics to designs.

1.2.2 MACROMECHANICS.

With the present state-of-the-art, the macromechanic approach [1] to the mechanics of filamentary composites is the most usable technique for the aerospace designer or stress analyst. The elastic constants and stress-strain response of an individual lamina may be determined experimentally, and these data may subsequently be used to determine the stress-strain response of a laminate composed of any orientation of the characterized laminae.

1.2.2.1 Lamina Constitutive Relationship.

For a filamentary composite (Fig. F1.2-1), the constituent laminae have three mutually perpendicular planes of elastic symmetry [1]. As discussed previously, a material with three mutually perpendicular planes of symmetry was termed orthotropic; therefore, the possibility exists to model the lamina as a homogeneous orthotropic medium. Since the thickness of an individual lamina is small relative to its other dimensions, it may be considered to be in a state of plane stress. The constitutive equation for the K-th lamina is then given by equation (F1.1-17) or

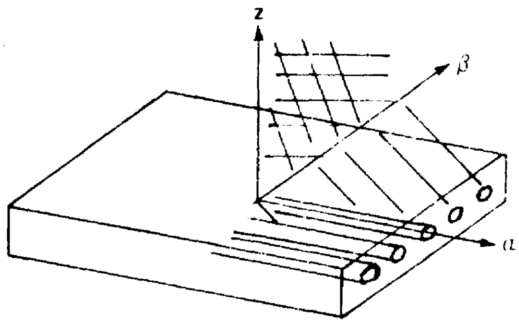


FIGURE F1.2-1. FILAMENTARY COMPOSITE WITH THREE MUTUALLY PERPENDICULAR PLANES OF SYMMETRY

$$\begin{bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \tau_{\alpha\beta} \end{bmatrix}^K = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & 2C_{66} \end{bmatrix}^K \begin{bmatrix} \epsilon_\alpha \\ \epsilon_\beta \\ \frac{1}{2}\gamma_{\alpha\beta} \end{bmatrix}^K . \qquad \text{(F1.2-1)}$$

The lamina stiffness matrix terms were defined in equation (F1.1-18) and are rewritten here as

$$C_{11} = \frac{E_{11}}{(1 - \nu_{12}\nu_{21})} ,$$

$$C_{22} = \frac{E_{22}}{(1 - \nu_{12}\nu_{21})} , \qquad \text{(F1.2-2)}$$

$$C_{12} = \frac{\nu_{21}E_{11}}{(1 - \nu_{12}\nu_{21})} = \frac{\nu_{12}E_{22}}{(1 - \nu_{12}\nu_{21})} \quad ,$$

and

$$C_{66} = G_{12} \quad .$$

As shown in equation (F1.1-8), the compliance matrix may be determined by inverting the stiffness matrix. This would result in

$$\begin{bmatrix} \epsilon_\alpha \\ \epsilon_\beta \\ \frac{1}{2}\gamma_{\alpha\beta} \end{bmatrix}^K = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & \frac{1}{2}S_{66} \end{bmatrix}^K \begin{bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \tau_{\alpha\beta} \end{bmatrix}^K \qquad \text{(F1. 2-3)}$$

where

$$S_{11} = \frac{1}{E_{11}} \quad ,$$

$$S_{22} = \frac{1}{E_{22}} \quad ,$$

$$S_{12} = -\frac{\nu_{12}}{E_{11}} = -\frac{\nu_{21}}{E_{22}} \quad , \qquad \text{(F1. 2-4)}$$

and

$$S_{66} = \frac{1}{G} \quad .$$

Since the lamina principal axes (α, β) generally do not coincide with the laminate reference axes (x, y) (Fig. 1.2-2), the stresses and strains for each lamina must be transformed as discussed previously. When this occurs, the constitutive relations for each lamina must also be transformed to the laminate reference axis system. The transformations, as discussed in paragraph F1.1.1, are

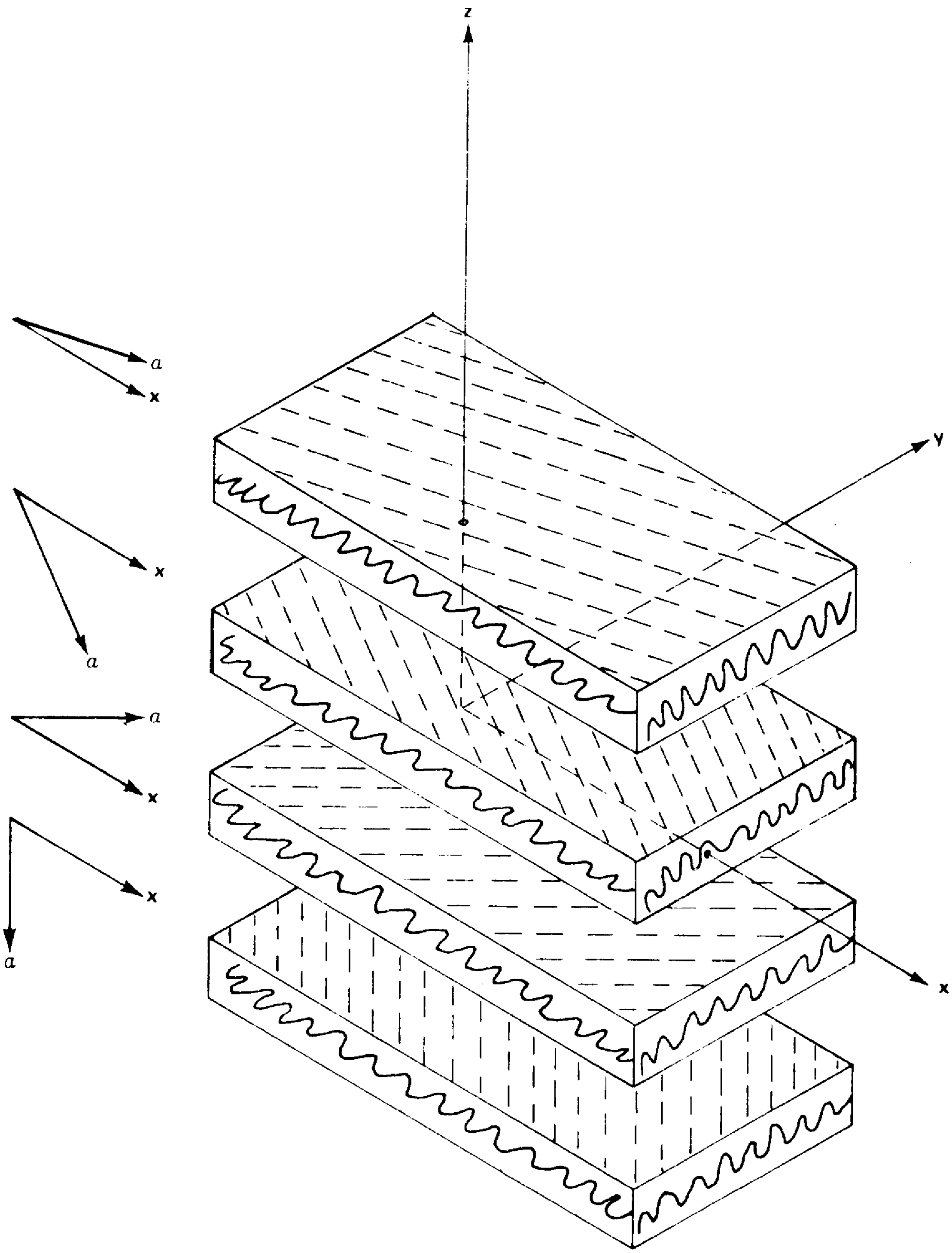


FIGURE F1.2-2. GENERAL LAMINAE ORIENTATION WITH LAMINATE REFERENCE AXIS

$$\begin{bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \tau_{\alpha\beta} \end{bmatrix}^K = [T] \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}^K \qquad \text{(F1. 2-5)}$$

and

$$\begin{bmatrix} \epsilon_\alpha \\ \epsilon_\beta \\ \frac{1}{2}\gamma_{\alpha\beta} \end{bmatrix}^K = [T] \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \frac{1}{2}\gamma_{xy} \end{bmatrix}^K \qquad \text{(F1.2-6)}$$

where [T] is defined in equation (F1. 1-3) and K denotes the K-th lamina.

Then

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}^K = [T]^{-1} \begin{bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \tau_{\alpha\beta} \end{bmatrix}^K \quad . \qquad \text{(F1. 2-7)}$$

The transformation matrix, T, may be written in a shortened form as

$$[T] = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \qquad \text{(F1. 2-8)}$$

where

$$m = \cos\theta$$

and

$$n = \sin\theta \quad .$$

Note that the inverse of the T matrix, $[T]^{-1}$, may be obtained by substituting for the positive angle θ a negative angle θ (refer to Fig. 1.2-2).

Using equations (F1.2-1), (F1.2-6), and (F1.2-7), the lamina constitutive equation, when transformed to the laminate reference axes, is

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}^K = [T]^{-1} [C']^K [T] \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \frac{1}{2}\gamma_{xy} \end{bmatrix}^K . \qquad \text{(F1.2-9)}$$

The transformed lamina stiffness matrix $[\overline{C}']$ is defined as

$$[\overline{C}']^K = [T]^{-1} [C']^K [T] = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & 2\overline{C}_{16} \\ \overline{C}_{12} & \overline{C}_{22} & 2\overline{C}_{26} \\ \overline{C}_{16} & \overline{C}_{26} & 2\overline{C}_{66} \end{bmatrix} \qquad \text{(F1.2-10)}$$

where the terms $\overline{C}_{ij}$ are given by equation (F1.1-29). The $\overline{C}'$ matrix, which is now fully populated ($C_{16} \neq C_{26} \neq 0$), appears to have six elastic constants which govern the lamina behavior; however, $\overline{C}_{16}$ and $\overline{C}_{26}$ are not independent as they are linear combinations of the four basic elastic constants. In the transformed coordinate system, the $\overline{C}'$ matrix is similar in appearance to the C matrix for a fully anisotropic lamina ($\overline{C}_{16} \neq 0$ and $\overline{C}_{26} \neq 0$), and the lamina is said to be "generally" orthotropic. Therefore, equation (F1.2-9) is said to be the constitutive equation for a "generally" orthotropic lamina.

Equation (F1. 2-1) is referred to as the constitutive equation for a "specially" orthotropic lamina ($C_{16} = C_{26} = 0$).

For convenience, equation (F1. 1-29) may be written as

$$\begin{aligned}
\bar{C}_{11} &= (3J_1 + J_2) + J_3 \cos 2\theta + J_4 \cos 4\theta, \\
\bar{C}_{22} &= (3J_1 + J_2) - J_3 \cos 2\theta + J_4 \cos 4\theta, \\
\bar{C}_{12} &= (J_1 - J_2) - J_4 \cos 4\theta, \qquad\qquad \text{(F1. 2-11)} \\
\bar{C}_{66} &= (J_1 + J_2) - J_4 \cos 4\theta, \\
\bar{C}_{16} &= \tfrac{1}{2} J_3 \sin 2\theta + J_4 \sin 4\theta,
\end{aligned}$$

and

$$\bar{C}_{26} = \tfrac{1}{2} J_3 \sin 2\theta - J_4 \sin 4\theta$$

where

$$\begin{aligned}
J_1 &= \tfrac{1}{8} [C_{11} + C_{22} + 2C_{12}] , \\
J_2 &= \tfrac{1}{2} [C_{66} - C_{12}] , \qquad\qquad \text{(F1. 2-12)} \\
J_3 &= \tfrac{1}{2} [C_{11} - C_{22}] ,
\end{aligned}$$

and

$$J_4 = \tfrac{1}{8} [C_{11} + C_{22} - 2C_{12} - 4C_{66}] .$$

Note that $\bar{C}_{11}$, $\bar{C}_{22}$, $\bar{C}_{12}$, and $\bar{C}_{66}$ are composed of a term independent of the angle of rotation (θ) and a term dependent on the angle of rotation. Therefore, it is evident that there are certain inherent lamina properties which are only dependent on the material being used.

1.2.2.2 Laminate Constitutive Relationship.

From Reference 1, for the formulation of a mathematical model of a laminate composed of orthotropic laminae, certain assumptions regarding the interaction between adjacent laminae must be made. Since practical uses of a laminate will normally dictate that it be thin relative to its other dimensions, the Kirchhoff-Love hypothesis (stresses perpendicular to the middle surface may be neglected, and the line segments originally normal to the middle surface remain straight and normal to the deformed surface and suffer neither extensions nor contractions) used in thin-plate and shell theory appear reasonable. Essentially, these assumptions reduce to

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}^K = \begin{bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{bmatrix} - z \begin{bmatrix} \chi_x \\ \chi_y \\ 2\chi_{xy} \end{bmatrix} \tag{F1.2-13}$$

where ϵ_x^o, ϵ_y^o, and γ_{xy}^o are the strains at the geometric middle surface of the laminate, and the χ's are the middle surface curvature. The transformed lamina constitutive equation, similar to equation (F1.2-9), is then

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [\bar{C}]^K \begin{bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{bmatrix} - z[\bar{C}]^K \begin{bmatrix} \chi_x \\ \chi_y \\ 2\chi_{xy} \end{bmatrix} \tag{F1.2-14}$$

where

$$[\overline{C}] = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{16} \\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{26} \\ \overline{C}_{16} & \overline{C}_{26} & \overline{C}_{66} \end{bmatrix} . \qquad \text{(F1. 2-15)}$$

This equation then relates the stress in the K-th lamina, oriented to the laminate reference axis, to the laminate middle surface strains and curvatures.

On an element of the laminate, the stress resultants and stress couples are defined as

$$N_x = \int_{-t/2}^{t/2} \sigma_x dz ,$$

$$N_y = \int_{-t/2}^{t/2} \sigma_y dz , \qquad \text{(F1. 2-16)}$$

and

$$N_{xy} = \int_{-t/2}^{t/2} \tau_{xy} dz$$

and

$$M_x = \int_{-t/2}^{t/2} z\sigma_x dz ,$$

$$M_y = \int_{-t/2}^{t/2} z\sigma_y dz , \qquad \text{(F1. 2-17)}$$

and

$$M_{xy} = \int_{-t/2}^{t/2} z\tau_{xy} dz \quad .$$

These integrals may be evaluated by integrating over each lamina and summing the results of each integration. By substituting equation (F1. 2-14) into equations (F1. 2-16) and (F1. 2-17),

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{bmatrix} - \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \chi_x \\ \chi_y \\ 2\chi_{xy} \end{bmatrix} \qquad \text{(F1. 2-18)}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{bmatrix} - \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \chi_x \\ \chi_y \\ 2\chi_{xy} \end{bmatrix} \qquad \text{(F1. 2-19)}$$

where

$$[A_{ij}] = \sum_{K=1}^{n} [\bar{C}_{ij}]^K (h_K - h_{K-1}) \quad , \qquad \text{(F1. 2-20)}$$

$$[B_{ij}] = \frac{1}{2} \sum_{K=1}^{n} [\bar{C}_{ij}]^K (h^2{}_K - h^2{}_{K-1}) \quad , \qquad \text{(F1. 2-21)}$$

and

$$[D_{ij}] = \frac{1}{3} \sum_{K=1}^{n} [\bar{C}_{ij}]^K (h^3{}_K - h^3{}_{K-1}) \quad . \qquad \text{(F1. 2-22)}$$

Refer to Figure F1. 2-3.

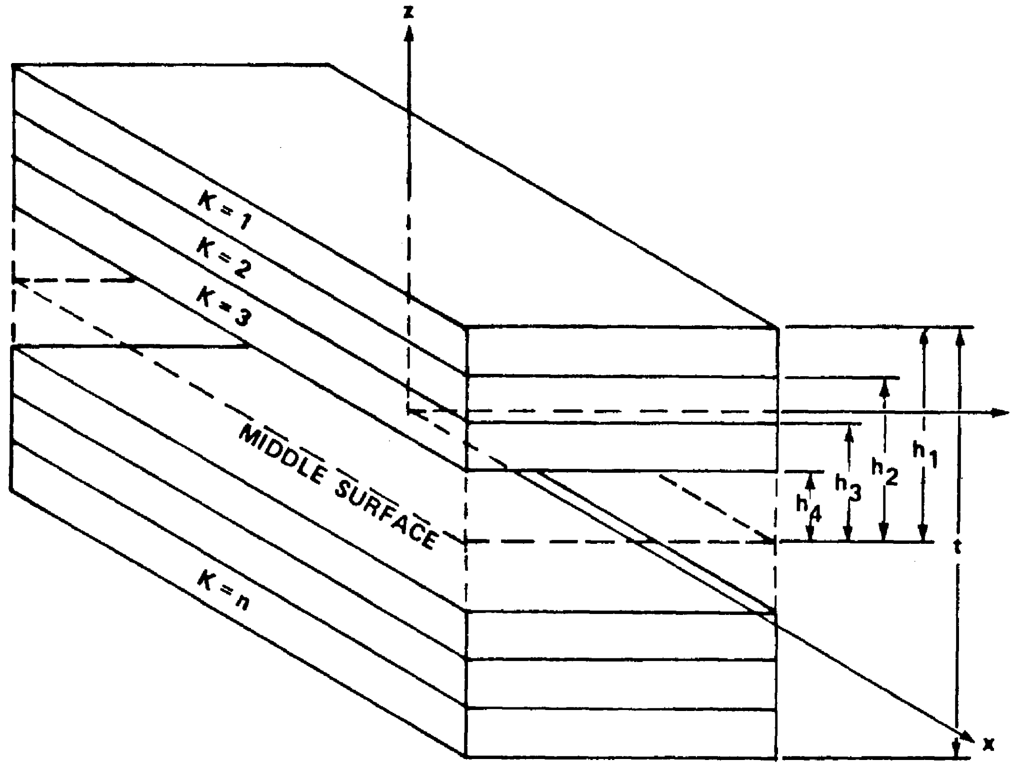


FIGURE F1.2-3. LAMINATE ELEMENT

Equations (F1.2-18) and (F1.2-19) are the constitutive equations for a laminated composite. They may be written as

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \epsilon^{o} \\ -\chi \end{bmatrix} \quad . \qquad (F1.2\text{-}23)$$

This is the general constitutive equation for laminated composites and is mathematically equivalent to the constitutive equation for a heterogeneous anisotropic medium. In this general form, the significant point is that there is coupling between extensional (membrane) deformation and bending deformation caused by the existence of the B matrix (Fig. F1-2.4). In other words, even within the limits of small deflection theory, forced curvatures within the laminate induce in-plane loads through this type of coupling. This coupling is caused by the neutral axis and the midplane of the laminate not being coincident.

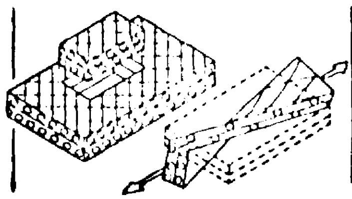


FIGURE F1.2-4. COUPLING OF DEFORMATION DUE TO B MATRIX

With various combinations of laminae, varying degrees of coupling may be caused. If the laminate is fabricated symmetric about the midplane (balanced), the B matrix will be identically zero, and the constitutive equation reduces to

$$[N] = [A] \ [\epsilon^o] \tag{F1.2-24}$$

and

$$[M] = -[D] \ [\chi] \quad . \tag{F1.2-25}$$

These equations are mathematically equivalent to the constitutive equations of a homogeneous anisotropic material. Hence, this type of laminate is referred to as homogeneous anisotropic (Fig. F1.2-5). At this stage the A and D matrices are fully populated and anisotropic in nature, and a second type of coupling still exists. For the A matrix the A_{16} and A_{26} terms couple the normal strains and shear stress, or the shear strains and normal stress.

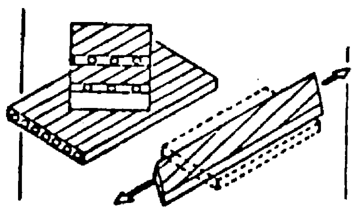


FIGURE F1.2-5.
COUPLING EFFECTS IN
A HOMOGENEOUS
ANISOTROPIC LAMINATE

For the D matrix the D_{16} and D_{26} terms couple the normal bending moments and twisting curvatures, and vice versa. If the laminate is symmetric about the x-y (laminate) axis (Fig. F1.2-6), this coupling may be reduced.

When the laminate has equal numbers of pairs of laminae symmetric about the x-y axis (termed angleply laminate),

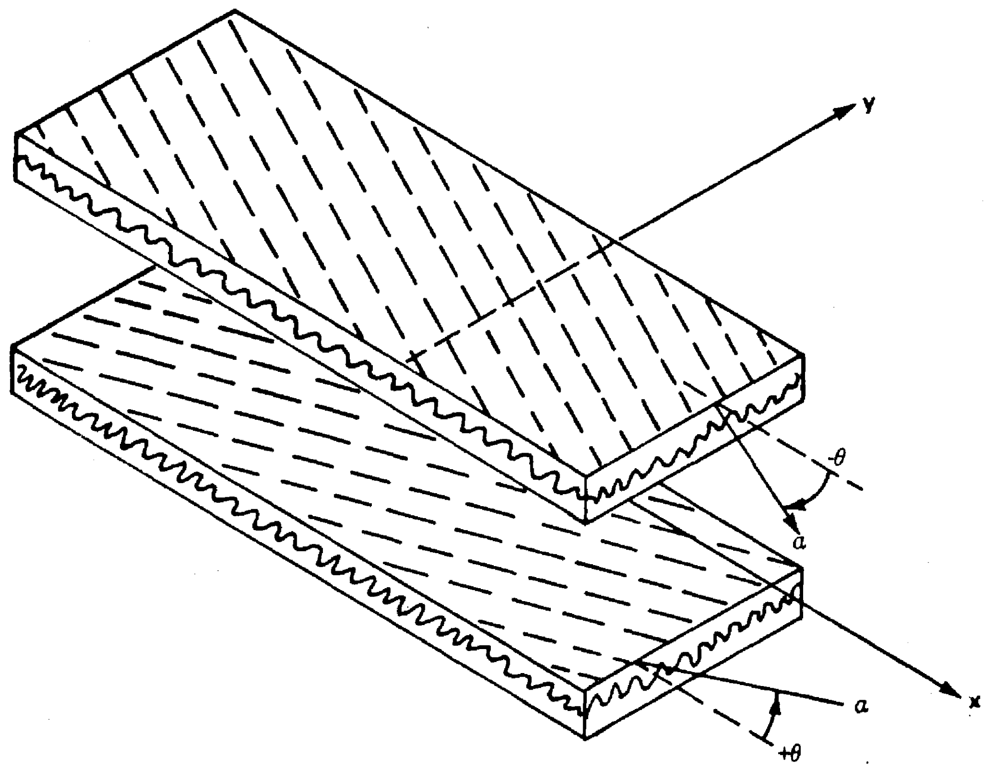


FIGURE F1.2-6. LAMINATE SYMMETRIC ABOUT THE x-y AXIS

the A matrix is orthotropic in nature ($A_{16} = A_{26} = 0$) (Fig. F1.2-7). The D matrix is still fully populated and anisotropic in nature. When the laminate has equal numbers of pairs of laminae at angles of 0 deg and 90 deg to the x-y axis (termed crossply laminate), the D matrix and the A matrix are orthotropic in nature.

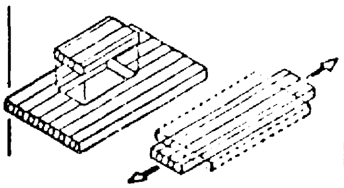


FIGURE F1.2-7. LAMINATE EXHIBITING ORTHOTROPIC CHARACTERISTICS

Because of the warpage which will occur during the fabrication process when symmetry does not exist, the laminates should be designed symmetric about the midplane, or nearly so.

Since most laminates are symmetric about the midplane, and because the majority of applications of advanced composites experience relatively low transverse shear strength, a reexamination of equation (F1.2-24) is warranted. Equation (F1.2-24) may be written as

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = [A] \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \qquad \text{(F1.2-26)}$$

when no bending occurs. Dividing both sides by the total laminate thickness yields

$$\begin{bmatrix} \overline{\sigma}_x \\ \overline{\sigma}_y \\ \overline{\tau}_{xy} \end{bmatrix} = \frac{1}{t} [A] \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} , \qquad \text{(F1.2-27)}$$

$$= \frac{1}{t} \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} ,$$

$$= \frac{1}{t} \begin{bmatrix} A_{11} & A_{12} & 2A_{16} \\ A_{12} & A_{22} & 2A_{26} \\ A_{16} & A_{26} & 2A_{66} \end{bmatrix} \cdot \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \frac{1}{2}\gamma_{xy} \end{bmatrix} ,$$

$$= [\bar{A}] \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \frac{1}{2}\gamma_{xy} \end{bmatrix} . \qquad \text{(F1. 2-28)}$$

The $\bar{\sigma}$'s are the average laminate stresses, and the $\bar{A}$ matrix may be defined as the laminate stiffness matrix. Thus,

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \frac{1}{2}\gamma_{xy} \end{bmatrix} = [A^*] \begin{bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\tau}_{xy} \end{bmatrix} \qquad \text{(F1. 2-29)}$$

where the laminate compliance matrix is

$$[A^*] = [\bar{A}]^{-1} = \begin{bmatrix} A_{11}^* & A_{12}^* & \frac{1}{2}A_{16}^* \\ A_{12}^* & A_{22}^* & \frac{1}{2}A_{26}^* \\ A_{16}^* & A_{26}^* & \frac{1}{2}A_{66}^* \end{bmatrix} . \qquad \text{(F1. 2-30)}$$

The gross or average laminate elastic moduli may be obtained from the laminate compliance matrix. For a balanced angleply laminate,

$$[A] = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} , \quad [\bar{A}] = \frac{1}{t} \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & 2A_{66} \end{bmatrix} ,$$

and

$$[A^*] = \begin{bmatrix} A_{11}^* & A_{12}^* & 0 \\ A_{12}^* & A_{22}^* & 0 \\ 0 & 0 & \frac{1}{2}A_{66}^* \end{bmatrix} . \qquad \text{(F1. 2-31)}$$

Then, comparing equation (F1. 2-29) with equation (F1. 2-3), the gross laminate elastic constants are

$$E_{xx} = \frac{1}{A_{11}^*} ,$$

$$E_{yy} = \frac{1}{A_{22}^*} ,$$

$$G_{xy} = \frac{1}{A_{66}^*} , \qquad \text{(F1. 2-32)}$$

$$\nu_{xy} = -\frac{A_{12}^*}{A_{11}^*} ,$$

$$\nu_{yx} = -\frac{A_{12}^*}{A_{22}^*} .$$

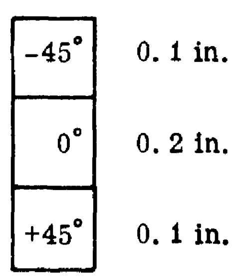
1.2.3 EXAMPLE CALCULATIONS.

The example problems which follow demonstrate some of the computations involved when working with composite materials.

1.2.3.1 <u>Example Problem 1.</u>

Calculate the A, B, and D matrices of the laminate constitutive equation for a three-ply laminate with the laminae oriented at - 45 deg, 0 deg, and +45 deg with the laminate axis (see the following sketch). The lamina material obeys

$$\begin{bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \tau_{\alpha\beta} \end{bmatrix} = 10^6 \begin{bmatrix} 30. & 1. & 0. \\ 1. & 3. & 0. \\ 0. & 0. & 1. \end{bmatrix} \begin{bmatrix} \epsilon_\alpha \\ \epsilon_\beta \\ \gamma_{\alpha\beta} \end{bmatrix} .$$



I. Example Problem 1 Laminate.

The [C] for the - 45-deg lamina and the + 45-deg lamina must be transformed to the laminate axes. Since the lamina material is homogeneous orthotropic, the values of the transformed stiffness matrix may be calculated using equation (F1.1-29) or equation (F1.2-11).

Using equation (F1.1-29), the $\overline{C}_{ij}$ terms of equation (F1.2-15) for the 45-deg lamina are

$$\overline{C}_{11} = 10^6 \left[30.\ \cos^4(-45) + 2(1. + 2.)\sin^2(-45)\cos^2(-45) + 3.\ \sin^4(-45) \right]$$
$$= 9.75 \times 10^6$$

$$\overline{C}_{26} = 10^6 \left[(30. - 1. - 2.)\sin^3(-45)\cos(-45) + (1. - 3. + 2.)\sin(-45)\cos^3(-45) \right]$$
$$= -6.75 \times 10^6 \ .$$

Then,

$$[\overline{C}]^{(1)} = 10^6 \begin{bmatrix} 9.75 & 7.75 & -6.75 \\ 7.75 & 9.75 & -6.75 \\ -6.75 & -6.75 & 7.75 \end{bmatrix} .$$

Similarly, for the +45-deg lamina

$$[\overline{C}]^{(3)} = 10^6 \begin{bmatrix} 9.75 & 7.75 & 6.75 \\ 7.75 & 9.75 & 6.75 \\ 6.75 & 6.75 & 7.75 \end{bmatrix} .$$

For the 0-deg lamina

$$[\overline{C}]^{(2)} = [C] = 10^6 \begin{bmatrix} 30. & 1. & 0. \\ 1. & 3. & 0. \\ 0. & 0. & 1. \end{bmatrix} .$$

From equation (F1.2-20),

$$A_{ij} = \sum_{K=1}^{n} (\overline{C}_{ij})^{(K)} (h_K - h_{K-1})$$
$$= 0.1\ [\overline{C}_{ij}]^{(1)} + 0.2\ [\overline{C}_{ij}]^{(2)} + 0.1\ [\overline{C}_{ij}]^{(3)} .$$

Thus,

$$A_{11} = 10^6 \,[9.75 \times 0.1 + 30. \times 0.2 + 9.75 \times 0.1] = 7.95 \times 10^6$$

$$\vdots$$

$$A_{66} = 10^6 \,[7.75 \times 0.1 + 1. \times 0.2 + 7.75 \times 0.1] = 1.75 \times 10^6 \;.$$

$$[A] = 10^6 \begin{bmatrix} 7.95 & 1.75 & 0. \\ 1.75 & 2.55 & 0. \\ 0. & 0. & 1.75 \end{bmatrix} \;.$$

From equation (F1.2-21)

$$B_{ij} = \tfrac{1}{2} \sum_{K=1}^{n} (\bar{C}_{ij})^{(K)} \; (h^2_K - h^2_{K-1})$$

$$= 0.015 \left[(\bar{C}_{ij})^{(3)} - (\bar{C}_{ij})^{(1)} \right] \;.$$

Thus,

$$B_{11} = 10^6 \times 0.015 \,[9.75 - 9.75] = 0$$

and

$$B_{66} = 0.$$

$$[B] = 10^6 \begin{bmatrix} 0. & 0. & 0.2025 \\ 0. & 0. & 0.2025 \\ 0.2025 & 0.2025 & 0. \end{bmatrix} \;.$$

From equation (F1.2-22)

$$D_{ij} = \frac{1}{3} \sum_{K=1}^{n} (\bar{C}_{ij})^{(K)} (h^3_K - h^3_{K-1})$$

$$= 0.00233 [\bar{C}_{ij}]^{(1)} + 0.00066 [\bar{C}_{ij}]^{(2)} + 0.00233 [\bar{C}_{ij}]^{(3)} .$$

Thus,

$$D_{11} = 10^6 [9.75 \times 0.00233 + 30. \times 0.00066 + 9.75 \times 0.00233]$$
$$= 0.06522 \times 10^6$$
$$\vdots$$
$$D_{66} = 10^6 [7.75 \times 0.00233 + 1. \times 0.00066 + 7.75 \times 0.00233]$$
$$= 0.03676 \times 10^6 .$$

$$[D] = 10^6 \begin{bmatrix} 0.06522 & 0.03676 & 0. \\ 0.03676 & 0.0474 & 0. \\ 0. & 0. & 0.03676 \end{bmatrix} .$$

Combining the results above, the constitutive equation may be written as

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = 10^6 \left[\begin{array}{ccc|ccc} 7.95 & 1.75 & 0. & 0. & 0. & 0.2025 \\ 1.75 & 2.55 & 0. & 0. & 0. & 0.2025 \\ 0. & 0. & 1.75 & 0.2025 & 0.2025 & 0. \\ \hline 0. & 0. & 0.2025 & 0.06522 & 0.03676 & 0. \\ 0. & 0. & 0.2025 & 0.03676 & 0.0474 & 0. \\ 0.2025 & 0.2025 & 0. & 0. & 0. & 0.03676 \end{array}\right] \begin{bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \\ \chi_x \\ \chi_y \\ 2\chi_{xy} \end{bmatrix} .$$

1.2.3.2 Example Problem 2.

Calculate the constitutive matrix for a four-ply laminate with laminae at + 45 deg, - 45 deg, - 45 deg, and + 45 deg and with a total laminate thickness of 0.4 in. (see the following sketch). Use the same lamina material as in paragraph 1.2.3.1.

+ 45°	0.1 in.
- 45°	0.1 in.
- 45°	0.1 in.
+ 45°	0.1 in.

II. Example Problem 2 Laminate.

Since the laminate is symmetric about the midplane and symmetric about the x-y axis, the constitutive equation will be in the form of

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & 0. \\ 0. & D \end{bmatrix} \begin{bmatrix} \epsilon^{o} \\ -\chi \end{bmatrix}$$

where

$$[A] = \begin{bmatrix} A_{11} & A_{12} & 0. \\ A_{12} & A_{22} & 0. \\ 0. & 0. & A_{66} \end{bmatrix}$$

and

$$[D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} .$$

Similar to paragraph 1.2.3.1,

$$[\bar{C}]^{(1)} = [\bar{C}]^{(4)} = 10^6 \begin{bmatrix} 9.75 & 7.75 & 6.75 \\ 7.75 & 9.75 & 6.75 \\ 6.75 & 6.75 & 7.75 \end{bmatrix}$$

and

$$[\bar{C}]^{(2)} = [\bar{C}]^{(3)} = 10^6 \begin{bmatrix} 9.75 & 7.75 & -6.75 \\ 7.75 & 9.75 & -6.75 \\ -6.75 & -6.75 & 7.75 \end{bmatrix} .$$

Using equations (F1.2-20) and (F1.2-22),

$$[A] = 10^6 \begin{bmatrix} 3.9 & 3.1 & 0. \\ 3.1 & 3.9 & 0. \\ 0. & 0. & 3.1 \end{bmatrix}$$

and

$$[D] = 10^6 \begin{bmatrix} 0.05187 & 0.04123 & 0.027 \\ 0.04123 & 0.05187 & 0.027 \\ 0.027 & 0.027 & 0.04123 \end{bmatrix} .$$

Therefore,

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = 10^6 \left[\begin{array}{ccc:ccc} 3.9 & 3.1 & 0. & & & \\ 3.1 & 3.9 & 0. & & 0. & \\ 0. & 0. & 3.1 & & & \\ \hdashline & & & 0.05187 & 0.04123 & 0.027 \\ & 0. & & 0.04123 & 0.05187 & 0.027 \\ & & & 0.027 & 0.027 & 0.04123 \end{array}\right] \begin{bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \\ \chi_x \\ \chi_y \\ 2\chi_{xy} \end{bmatrix} .$$

1.3 LAMINATE CODING.

In Reference 2 a laminate orientation code was devised for filamentary composites which provided both a concise reference and a positive identification of any laminate. As expressed in that reference, the following type of coding is intended to provide a means of achieving conciseness in engineering presentations and communications; however, it is neither recommended nor discouraged that this code be employed on shop drawings since that policy is an internal one which must be decided by each using organization.

1.3.1 STANDARD CODE ELEMENTS.

The formulation of the code must be adequate to specify as concisely as possible (1) the angles of laminae relative to a reference axis (the x-axis), (2) the number of laminae at each angle, and (3) the exact geometric sequence of laminae.

The basic laminate code will adhere to the following guidelines [2]:

1. Each lamina is denoted by a number representing its orientation in degrees between its filament direction (refer to Fig. F1.0-3) and the x-axis.

2. Individual adjacent laminae are separated by a slash if their angles are different.

3. The laminae are listed in sequence from one laminate face to the other, with brackets indicating the beginning and end of the code. The first lamina should be the most positive lamina in the z direction (refer to Fig. F1.2-3).

4. Adjacent laminae of the same angle are deonoted by a numerical subscript.

5. A subscript T to the bracket indicates that the total laminate is shown.

When adjacent laminae are of the same angle but opposite in sign, the appropriate use of + or - signs may be employed. Each + or - sign represents one lamina and supersedes the use of the numerical subscript, which is used only when the directions are identical. Note that positive angles are assumed to be counterclockwise.

Several examples are shown demonstrating the basic coding.

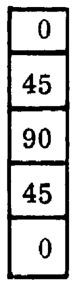
45
0
-60
-60
30

Code

$[45/0/-60_2/30]_T$

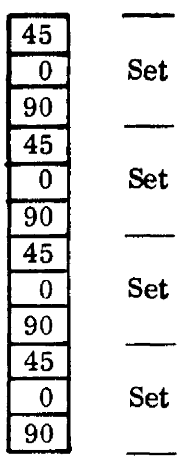
+45
-45
-30
+30
0
+45
+45
-45
-45

$[\pm 45/\mp 30/0/\pm(45_2)]_T$

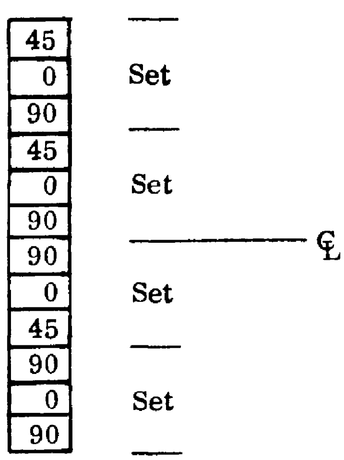
Laminate	Code
	$[0/45/\overline{90}]_S$

A repeating sequence of laminae is termed a set. A set is coded in accordance with the same rules which apply to a single lamina:

Laminate



$[(45/0/90)_4]_T$ or $[45/0/90]_{4T}$



$[(45/0/90)_2]_S$ or $[45/0/90]_{2S}$

1.4 COMPUTER PROGRAMS IN COMPOSITE ANALYSIS.

Several computer programs have been made available to MSFC and support personnel to assist in analyzing composite material elements. These programs have been (or are currently being) documented in References 4 and 5. The program names and a brief comment on each are shown in Table F1.4-1.

TABLE F1.4-1. Computer Programs in Composite Analysis

Program Name	Comment
CDLAMA	Yield analysis of composite plates composed of orthotropic lamina with in-plane loading.
GDLAMT	Yield analysis of composite plates composed of orthotropic lamina with in-plane loading.
GREPPA	Design composite skin/stringer/frame compression panel.
INTACT	Locates optimum strength envelopes for laminates under the action of combined loading.
LAMCHK	Laminate check - computes margins of safety for specified 0-deg, 90-deg, ± 45-deg boron epoxy laminates under combined loading.
LAP	Analysis of single overlap bonded joints with Metlbond 329 adhesive mechanical behavior.
LAPI	Analyzes single overlap bonded joints and accepts arbitrary applied loads and incorporates the B basis correction factor.
MMBCK	Calculates the buckling loads of radially inhomogeneous anisotropic, cylindrical shells wherein the effects of boundary conditions are not considered.
PANBUCK II	Panel buckling — calculates critical buckling loads and mode for orthotropically layered, rectangular, anisotropic plates and honeycomb sandwich panels. Also computes local instability modes of failure for composite panels.
STAB	Stability analysis — local instability analysis of orthotropic honeycomb panels, columns, and beams — failure-mode analysis for filament rupture, intercell dimpling, and layer instability.

1.5 REFERENCES.

1. Petit, P. H.: Basic Concepts in the Design and Stress Analysis of Laminated Composites. Report No. SMN 239, Lockheed-Georgia Co., division of Lockheed Aircraft Corp., Marietta, Ga., March 1968.

2. Structural Design Guide for Advanced Composite Applications. Advanced Composites Division, Air Force Materials Laboratory, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio, August 1969.

3. Ashton, J. C.; Halpin, J. C.; and Petit, P. H.: Primer on Composite Materials: Analysis. Technomic Publishing Co., Inc., Stamford, Conn., 1969.

4. Structural Analysis Computer Utilization Manual. Analytical Mechanics Division, Astronautics Laboratory, George C. Marshall Space Flight Center (NASA).

5. Strength Analysis Computer Program Description Manual. Analytical Mechanics Division, Astronautics Laboratory, George C. Marshall Space Flight Center (NASA).