II. Fixed-Fixed Beam.

[Diagram of a fixed-fixed beam with labeling A, B, and x]

A. Boundary Conditions:

\[ v = \frac{dv}{dx} = 0, \quad @ \quad x = 0, L, \]

\[ v(x) = -\int_{0}^{x} \int_{0}^{x_2} \left[ \frac{M_T(x_1) + M_{0_1} + x_1 V_{0_1}}{E I_z(x_1)} \right] dx_1 dx_2, \]

\[ v\left(\frac{x}{L}\right) = -\frac{c F_1}{I_{z_0}} L^2 \left( I_{1x} + F_2 I_{2x} + F_3 I_{3x} \right), \]

where

\[ I_{2x} = \int_{0}^{x} \int_{0}^{x_2} \frac{dx_1}{h(x_1) g'(x_1)} dx_2, \quad I_{2} = \int_{0}^{1} \int_{0}^{x_2} \frac{dx_1}{h(x_1) g'(x_1)} dx_2, \]

\[ I_{3x} = \int_{0}^{x} \int_{0}^{x_2} \frac{x dx dx_2}{h(x_1) g'(x_1)}, \quad I_{3} = \int_{0}^{1} \int_{0}^{x_2} \frac{x dx}{h(x_1) g'(x_1)} dx_2, \]

\[ F_2 = \frac{\int_{0}^{1} \frac{f(x_1)}{g'(x_1)} - I_1 \int_{0}^{1} \frac{x_1 dx_1}{h(x_1) g'(x_1)}}{I_{2} \int_{0}^{1} \frac{x_1 dx_1}{h(x_1) g'(x_1)} - I_3 \int_{0}^{1} \frac{dx_1}{h(x_1) g'(x_1)}}, \]
\[ F_3 = \frac{I_1 \int_0^1 \frac{dx_1}{h(x_1) g''(x_1)} - I_2 \int_0^1 \frac{f(x_1)}{g''(x_1)} \, dx_1}{I_2 \int_0^1 \frac{x_1 \, dx_1}{h(x_1) g''(x_1)} - I_3 \int_0^1 \frac{dx_1}{h(x_1) g''(x_1)}}. \]

(Table 3.0-4 gives values for \( F_2 \) and \( F_3 \))

\[ M_{0z} = \alpha E F_1 F_2, \]

\[ V_{0z} = \frac{\alpha}{L} E F_1 F_3, \]

\[ M_z(x) = M_{0z} + x V_{0z}, \]

\[ u_{av}(x) = \frac{\alpha F_0 L}{\Lambda_0} \int_0^x f(x_1) \, dx_1. \]

If the end \( B \) is restrained against longitudinal motion, then

\[ u_{av}(x) = 0, \]

\[ \sigma_{xx} = -\alpha ET \left( \frac{M_T}{y} + \frac{M_y}{L} \right) z + \left( \frac{M_{Tz}}{I_z} + \frac{M_z}{I_y} \right) y. \]

\[ v(x), M_0, V_0 \] are same as above.

**CASE a:** \( EI_z(x) = \text{constant, and } f(x) = \left( \frac{X}{L} \right)^q \)

\[ v \left( \frac{X}{L} \right) = \frac{\alpha F_1 x^2}{l_z (q+1) (q+2)} \left[ \frac{x^q}{L^q} + q \left( 1 - \frac{X}{L} \right)^q - 1 \right], \]

\[ M_{0z} = \frac{2 \alpha EF_1 (q-1)}{(q+1) (q+2)}. \]
### TABLE 3.0-4. VALUES OF CONSTANTS F₂, F₃, AND F₄

<table>
<thead>
<tr>
<th>G</th>
<th>F₂</th>
<th>F₃</th>
<th>F₄</th>
<th>G</th>
<th>F₂</th>
<th>F₃</th>
<th>F₄</th>
<th>G</th>
<th>F₂</th>
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<th>F₄</th>
<th>G</th>
<th>F₂</th>
<th>F₃</th>
<th>F₄</th>
</tr>
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<td>0</td>
<td>1.000</td>
<td>0.000</td>
<td>-1.000</td>
<td>0</td>
<td>1.000</td>
<td>0.000</td>
<td>-1.000</td>
<td>0</td>
<td>1.000</td>
<td>0.000</td>
<td>-1.000</td>
<td>0</td>
<td>1.000</td>
<td>0.000</td>
<td>-1.000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.999</td>
<td>-0.000</td>
<td>-0.999</td>
<td>0.1</td>
<td>0.999</td>
<td>-0.000</td>
<td>-0.999</td>
<td>0.1</td>
<td>0.999</td>
<td>-0.000</td>
<td>-0.999</td>
<td>0.1</td>
<td>0.999</td>
<td>-0.000</td>
<td>-0.999</td>
</tr>
<tr>
<td>0.2</td>
<td>0.989</td>
<td>-0.010</td>
<td>-0.979</td>
<td>0.2</td>
<td>0.989</td>
<td>-0.010</td>
<td>-0.979</td>
<td>0.2</td>
<td>0.989</td>
<td>-0.010</td>
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<td>0.2</td>
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<td>-0.010</td>
<td>-0.979</td>
</tr>
<tr>
<td>0.3</td>
<td>0.979</td>
<td>-0.020</td>
<td>-0.959</td>
<td>0.3</td>
<td>0.979</td>
<td>-0.020</td>
<td>-0.959</td>
<td>0.3</td>
<td>0.979</td>
<td>-0.020</td>
<td>-0.959</td>
<td>0.3</td>
<td>0.979</td>
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<td>-0.959</td>
</tr>
<tr>
<td>0.4</td>
<td>0.969</td>
<td>-0.030</td>
<td>-0.949</td>
<td>0.4</td>
<td>0.969</td>
<td>-0.030</td>
<td>-0.949</td>
<td>0.4</td>
<td>0.969</td>
<td>-0.030</td>
<td>-0.949</td>
<td>0.4</td>
<td>0.969</td>
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<td>-0.949</td>
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<td>-0.939</td>
<td>0.5</td>
<td>0.959</td>
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<td>0.959</td>
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<td>-0.929</td>
<td>0.6</td>
<td>0.949</td>
<td>-0.050</td>
<td>-0.929</td>
<td>0.6</td>
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<td>-0.050</td>
<td>-0.929</td>
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<td>-0.919</td>
<td>0.7</td>
<td>0.939</td>
<td>-0.060</td>
<td>-0.919</td>
<td>0.7</td>
<td>0.939</td>
<td>-0.060</td>
<td>-0.919</td>
<td>0.7</td>
<td>0.939</td>
<td>-0.060</td>
<td>-0.919</td>
</tr>
<tr>
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<td>0.929</td>
<td>-0.070</td>
<td>-0.909</td>
<td>0.8</td>
<td>0.929</td>
<td>-0.070</td>
<td>-0.909</td>
<td>0.8</td>
<td>0.929</td>
<td>-0.070</td>
<td>-0.909</td>
<td>0.8</td>
<td>0.929</td>
<td>-0.070</td>
<td>-0.909</td>
</tr>
<tr>
<td>0.9</td>
<td>0.919</td>
<td>-0.080</td>
<td>-0.899</td>
<td>0.9</td>
<td>0.919</td>
<td>-0.080</td>
<td>-0.899</td>
<td>0.9</td>
<td>0.919</td>
<td>-0.080</td>
<td>-0.899</td>
<td>0.9</td>
<td>0.919</td>
<td>-0.080</td>
<td>-0.899</td>
</tr>
<tr>
<td>1.0</td>
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<td>-0.090</td>
<td>-0.889</td>
<td>1.0</td>
<td>0.909</td>
<td>-0.090</td>
<td>-0.889</td>
<td>1.0</td>
<td>0.909</td>
<td>-0.090</td>
<td>-0.889</td>
<td>1.0</td>
<td>0.909</td>
<td>-0.090</td>
<td>-0.889</td>
</tr>
</tbody>
</table>

### Notes:

\[ h(x_4) = 1 + H \left( \frac{x_4}{L} \right) \]

\[ g(x_4) = 1 + G \left( \frac{x_4}{L} \right) \]
\[ V_{0z} = - \frac{6 \alpha E F_1 q}{(q+1)(q+2)L} \]

\[ M_z = \frac{2 \alpha E F_1}{(q+1)(q+2)} \left[ \left(1 - 3 \frac{x}{L}\right) q - 1 \right] \]

**CASE b:**

\[ \frac{M_{T_z}(x)}{EI_z(x)} = \text{constant} \]

\[ v(x) = 0 \]

\[ M_{0z} = -M_{T_z} \]

\[ V_{0z} = 0 \quad M_z = -M_{T_z} = \text{constant} \]

**III. Fixed-Hinged Beam.**

**A. Boundary Conditions:**

\[ v = \frac{dv}{dx} = 0, \quad @ \ x = 0 \]

\[ v = \frac{d^2v}{dx^2} + \frac{M_{T_z}}{EI_z} = 0, \quad @ \ x = L \]
\[ v \left( \frac{x}{L} \right) = -\frac{\alpha F_4 L^2}{I z_0} \left[ I_{x_1} + F_4 \left( I_{x_2} - I_{x_3} \right) \right] , \]

where

\[ F_4 = \frac{I_4}{(I_3 - I_3)} \quad \text{(refer to Table 3.0-4 for values of } F_4), \]

\[ V_{0z} = -\frac{\alpha E F_1 F_4}{L} , \]

\[ M_{0z} = \alpha E F_1 F_4 , \]

\[ u_{av}(x) = \frac{\alpha F_0 L}{A_0} \int_0^{x_1} f(x_1) \, dx_1 , \]

\[ \sigma_{xx} = -\alpha ET + \frac{P_T}{A} + \left( \frac{M_T + M_z}{I_z} \right) y + \left( \frac{M_T + M_y}{I_y} \right) z . \]

If the end of B is hinged,

\[ u_{av}(x) = 0 , \]

\[ \sigma_{xx} = -\alpha ET \left( \frac{M_T + M_z}{I_z} \right) y + \left( \frac{M_T + M_y}{I_y} \right) z . \]

**CASE a:**

\[ EI_z = \text{constant} , \]

\[ f(x_1) = \left( \frac{x}{L} \right)^q , \]
\[ v \left( \frac{x}{L} \right) = -\alpha \frac{F_1 L^2}{I_{0z}} \left( \left( \frac{x}{L} \right)^{q+2} - \frac{1}{2} \left[ 3 \left( \frac{x}{L} \right)^2 - \left( \frac{x}{L} \right)^3 \right] \right) \frac{1}{(q+1)(q+2)} \]

\[ V_{0z} = \frac{3 \alpha E F_1}{(q+1)(q+2)} \frac{L}{L} \]

\[ M_{0z} = -\frac{3 \alpha E F_1}{(q+1)(q+2)} \]

\[ M_z = -\frac{3 \alpha E F_1 \left( 1 - \frac{x}{L} \right)}{(q+1)(q+2)} \]

**CASE b:**

\[ M^*_z = \frac{M_z}{E I_z} (x) = \text{constant} \]

\[ v(x) = \frac{\alpha F_1}{I_z} \frac{x^2}{4} \left( 1 - \frac{x}{L} \right) \]

\[ V_{0z} = \frac{3 \alpha E F_1}{L} \]

\[ M_{0z} = -\frac{3}{2} \alpha E F_1 \]

\[ M_z = -\frac{3}{2} \left( 1 - \frac{x}{L} \right) \alpha E F_1 \]

**IV. Deflection Plots.**

From the previous cases the deflection expressions were found to be

**A. Simply Supported Beam:**

\[ v(x) = \frac{F_1 L^2}{I_{0z}} \left( \frac{x}{L} \left( I_1 - I_{1x} \right) \right) \]
B. Fixed-Fixed Beam:

\[ v(x) = -\frac{F_1 L^2}{I_{0z}} \left( I_{1x} + F_2 I_{2x} + F_3 I_{3x} \right) \]

C. Fixed-Hinged Beam:

\[ v(x) = -\frac{F_1 L^2}{I_{0z}} \left[ I_{1x} + F_4 \left( I_{2x} - I_{3x} \right) \right] \]

These expressions have been plotted for parameter variations of beam width, depth, and degree of thermal gradient along the beam length. The plots are made in nondimensional form with the following designation:

\[ SS = \left( \frac{X}{L} \right) \left( I_{1x} - I_{1x} \right) \]

\[ FS = - \left[ I_{1x} + F_4 \left( I_{2x} - I_{3x} \right) \right] \]

and

\[ FF = - \left( I_{1x} + F_2 I_{2x} + F_3 I_{3x} \right) \]

Therefore the deflection, \( v(x) \), for any case can be found by multiplying \( SS \), \( FS \), or \( FF \) by \( -\frac{F_1 L^2}{I_{0z}} \).

Figures 3.0-4 and 3.0-5 show the variation of \( FF \) and \( FS \) as a function of lengthwise temperature gradient. In the figure, \( G = H = 0 \) (constant cross section) and \( N \) equals the exponent of the thermal variation along the length of the beam; e.g., \( N = 0 \) means constant variation, \( N = 1 \) means linear variation, etc.

Figures 3.0-6 through 3.0-13 show the three deflection parameters \( SS \), \( FS \), and \( FF \) for variation of parameters \( G \), \( H \), and \( N \), where \( H \) equals variation in width of beam along the length, \( G \) equals variation in depth at \( x = 0 \) (refer to Paragraph 3.0.3.1).
Figure 3.0-4. Deflection parameter FF versus distance along beam for variation of lengthwise temperature gradient.
Figure 3.0-5. Deflection parameter FS versus distance along beam for variation of lengthwise temperature gradient.
Figure 3.0-6. Values of SS, FS, FF versus values of $T(x/L)$ for $H = 0$, $G = 0$, $N = 1$. 
Figure 3.0-7. Values of SS, FS, FF versus values of T for H = 0, G = 0, N = 2.
Figure 3.0-8. Values of SS, FS, FF versus values of T for $H = 0, \ G = 0.5, \ N = 1$. 
Figure 3.0-9. Values of SS, FS, FF versus values of T for $H = 0$, $G = 0.5$, $N = 2$. 
Figure 3.0-10. Values of SS, FS, FF versus values of T for $H = 0$, $G = 1.0$, $N = 1$. 
Figure 3.0-11. Values of SS, FS, FF versus values of T for H = 0, G = 1.0, N = 2.
Figure 3.0-12. Values of SS, FS, FF versus values of T for $H = 0$, $G = 2.0$, $N = 2$. 
Figure 3.0-13. Values of SS, FS, FF versus values of T for $H = 0, \ G = 2.0, \ N = 2$. 
3.0.2.3 Representation of Temperature Gradient by Polynomial.

A temperature profile obtained analytically or experimentally can be approximated by a polynomial of the form

\[ T = \sum_{n=1}^{N} \sum_{m=1}^{M} V_{mn} y^n z^m + T_0. \]

Accuracy of this approximation increases with the increase in the number of terms of the polynomial; but, since the temperature distribution is to be integrated in the process of obtaining the stress distribution, the accuracy of the resulting distribution using a low-order polynomial improves. The total number of grid points required to determine the coefficients \( V_{mn} \) uniquely is equal to \((M+N)\).

The coefficients \( V_{mn} \) are obtained from the following relations.

\[
\begin{align*}
V_{11} &= [a_{1,11}, a_{1,12}, a_{1,13}, \ldots, a_{1,1N}, a_{1,21}, \ldots, a_{1,MN}]^T - T_0 \\
V_{12} &= [a_{2,11}, a_{2,12}, a_{2,13}, \ldots, a_{2,1N}, a_{2,21}, \ldots, a_{2,MN}]^T - T_0 \\
V_{1N} &= [a_{N,11}, a_{N,12}, \ldots, a_{N,1N}, a_{N,21}, \ldots, a_{N,MN}]^T - T_0 \\
V_{21} &= [a_{11}, a_{12}, a_{13}, \ldots, a_{1N}, a_{21}, \ldots, a_{MN}]^T - T_0 \\
V_{22} &= [a_{21}, a_{22}, a_{23}, \ldots, a_{2N}, a_{21}, \ldots, a_{MN}]^T - T_0 \\
V_{2N} &= [a_{(M+N),11}, a_{(M+N),12}, \ldots, a_{(M+N),1N}, a_{(M+N),21}, \ldots, a_{(M+N),MN}]^T - T_0.
\end{align*}
\]
where \( a_{i,mn} = y_i^n z_i^m \). The values \( (y_i, z_i) \) are coordinates of the \( i \)th grid point at temperature \( T_i \) and \( T_0 \) is temperature at the centroid of the cross section.

If the temperature difference \( (T_i - T_0) \) is continuous and symmetric about the \( z \)-axis, then \( V_{m1} = V_{m3} = V_{m5} = \ldots = 0 \), and if it is continuous and antisymmetric \( V_{m2} = V_{m4} = V_{m6} = \ldots = 0 \). Similarly, for temperature differences that are symmetric about the \( y \)-axis, \( V_{n1} = V_{n3} = V_{n5} = \ldots = 0 \), and if it is antisymmetric about the \( y \)-axis, \( V_{n2} = V_{n4} = V_{n6} = \ldots = 0 \).

If the temperature varies only in one direction (e.g., for variation along the \( y \)-direction only) along either the \( y \)- or \( z \)-axis, the polynomial reduces to

\[
T_i - T_0 = \sum_{n=1}^{N} V_n y^n.
\]

\[
\begin{pmatrix}
V_1 \\
V_2 \\
\vdots \\
V_n
\end{pmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\begin{pmatrix}
T_1 - T_0 \\
T_2 - T_0 \\
\vdots \\
T_n - T_0
\end{pmatrix}.
\]

\[
a_{in} = y_i^n
\]

As before, if the temperature difference is continuous and symmetric, \( V_1 = V_3 = V_5 = \ldots = 0 \); and for continuous antisymmetric temperature differences \( V_2 = V_4 = V_6 = \ldots = 0 \).

If the grid points are equally spaced on the cross section, say at distance \( d_1 \), then the polynomial can be written in nondimensional form as

\[
T_i - T_0 = \sum_{n=1}^{N} V_n d_1^n \left( \frac{y}{d_1} \right)^n.
\]
EXAMPLES:

1. Continuous Symmetric Temperature

   a. Nine Points:

   $T_i - T_0 = \sum_{n=2, 4, 6} (V_n \, d_i^n) \, (i)^n$

   \[
   \begin{bmatrix}
   V_2 \, d_1^2 \\
   V_4 \, d_1^4 \\
   V_6 \, d_1^6 \\
   V_8 \, d_1^8 \\
   \end{bmatrix} =
   \begin{bmatrix}
   1^2 & 1^4 & 1^6 & 1^8 \\
   2^2 & 2^4 & 2^6 & 2^8 \\
   3^2 & 3^4 & 3^6 & 3^8 \\
   4^2 & 4^4 & 4^6 & 4^8 \\
   \end{bmatrix}^{-1}
   \begin{bmatrix}
   T_1 - T_0 \\
   T_2 - T_0 \\
   T_3 - T_0 \\
   T_4 - T_0 \\
   \end{bmatrix}
   \]

   \[
   \begin{bmatrix}
   0.16000+01 \\
   -0.67778+00 \\
   0.80556-01 \\
   -0.27778-02 \\
   \end{bmatrix}
   =
   \begin{bmatrix}
   0.16000+01 & -0.20000+00 & 0.25397-01 & -0.17857-02 \\
   -0.67778+00 & 0.23472+00 & -0.33333-01 & 0.24306-02 \\
   0.80556-01 & 0.36111-01 & 0.83333-02 & -0.69444-03 \\
   -0.27778-02 & 0.13889-02 & -0.39683-03 & 0.49603-04 \\
   \end{bmatrix}
   \begin{bmatrix}
   T_1 - T_0 \\
   T_2 - T_0 \\
   T_3 - T_0 \\
   T_4 - T_0 \\
   \end{bmatrix}
   \]

   b. Seven Points:

   \[
   \begin{bmatrix}
   V_2 \, d_1^2 \\
   V_4 \, d_1^4 \\
   V_6 \, d_1^6 \\
   \end{bmatrix} =
   \begin{bmatrix}
   1^2 & 1^4 \\
   2^2 & 2^4 \\
   3^2 & 3^4 \\
   4^2 & 4^4 \\
   \end{bmatrix}^{-1}
   \begin{bmatrix}
   T_1 - T_0 \\
   T_2 - T_0 \\
   T_3 - T_0 \\
   \end{bmatrix}
   \]

   \[
   \begin{bmatrix}
   0.15000+01 \\
   -0.54167+00 \\
   0.41667-01 \\
   \end{bmatrix}
   =
   \begin{bmatrix}
   0.15000+01 & -0.15000+00 & 0.11111-01 \\
   -0.54167+00 & 0.16667+00 & -0.13889-01 \\
   0.41667-01 & 0.16667-01 & 0.27778-02 \\
   \end{bmatrix}
   \begin{bmatrix}
   T_1 - T_0 \\
   T_2 - T_0 \\
   T_3 - T_0 \\
   \end{bmatrix}
   \]

   c. Five Points:

   \[
   \begin{bmatrix}
   V_2 \, d_1^2 \\
   V_4 \, d_1^4 \\
   \end{bmatrix} =
   \begin{bmatrix}
   1^2 & 1^4 \\
   2^2 & 2^4 \\
   \end{bmatrix}
   \begin{bmatrix}
   T_1 - T_0 \\
   T_2 - T_0 \\
   \end{bmatrix}
   \]

   \[
   \begin{bmatrix}
   0.15000+01 \\
   -0.54167+00 \\
   0.41667-01 \\
   \end{bmatrix}
   =
   \begin{bmatrix}
   4 & -1 \\
   3 & -1 \\
   \end{bmatrix}
   \begin{bmatrix}
   T_1 - T_0 \\
   T_2 - T_0 \\
   \end{bmatrix}
   \]

   \[
   \begin{bmatrix}
   0.15000+01 \\
   -0.54167+00 \\
   0.41667-01 \\
   \end{bmatrix}
   =
   \begin{bmatrix}
   4 & -1 \\
   3 & -1 \\
   \end{bmatrix}
   \begin{bmatrix}
   T_1 - T_0 \\
   T_2 - T_0 \\
   \end{bmatrix}
   \]

   \[
   \begin{bmatrix}
   0.15000+01 \\
   -0.54167+00 \\
   0.41667-01 \\
   \end{bmatrix}
   =
   \begin{bmatrix}
   4 & -1 \\
   3 & -1 \\
   \end{bmatrix}
   \begin{bmatrix}
   T_1 - T_0 \\
   T_2 - T_0 \\
   \end{bmatrix}
   \]

   \[
   \begin{bmatrix}
   0.15000+01 \\
   -0.54167+00 \\
   0.41667-01 \\
   \end{bmatrix}
   =
   \begin{bmatrix}
   4 & -1 \\
   3 & -1 \\
   \end{bmatrix}
   \begin{bmatrix}
   T_1 - T_0 \\
   T_2 - T_0 \\
   \end{bmatrix}
   \]
2. Continuous Antisymmetric Temperature

a. Nine Points:

\[
\begin{align*}
V_1 d_1 & = \begin{bmatrix} 0.15003+01 & -0.30031+00 & 0.33466-01 & -0.22126-04 \end{bmatrix} T_1-T_0 \\
V_3 d_1^3 & = \begin{bmatrix} -0.54209+00 & 0.33375+00 & -0.41847-01 & 0.30116-04 \end{bmatrix} T_2-T_0 \\
V_5 d_1^5 & = \begin{bmatrix} 0.41787-01 & -0.33454-01 & 0.83850-02 & -0.86045-05 \end{bmatrix} T_3-T_0 \\
V_7 d_1^7 & = \begin{bmatrix} -0.86045-05 & 0.86045-05 & -0.36876-05 & 0.61461-06 \end{bmatrix} T_4-T_0
\end{align*}
\]

b. Seven Points:

\[
\begin{align*}
V_1 d_1 & = \begin{bmatrix} 0.15000+01 & -0.30000+00 & 0.33333-01 \end{bmatrix} T_1-T_0 \\
V_3 d_1^3 & = \begin{bmatrix} -0.54167+00 & 0.33333+00 & -0.41667-01 \end{bmatrix} T_2-T_0 \\
V_5 d_1^5 & = \begin{bmatrix} 0.41667-01 & -0.33333-01 & 0.83333-02 \end{bmatrix} T_3-T_0
\end{align*}
\]

c. Five Points:

\[
\begin{align*}
V_1 d_1 & = \begin{bmatrix} 1 & 1^3 \end{bmatrix}^{-1} T_1-T_0 = \begin{bmatrix} 4/3 & -1/3 \end{bmatrix} T_1-T_0 \\
V_3 d_1^3 & = \begin{bmatrix} 2 & 2^3 \end{bmatrix} T_2-T_0 = \begin{bmatrix} -1/6 & 1/6 \end{bmatrix} T_2-T_0
\end{align*}
\]

d. Three Points:

\[
V_1 d_1 = T_1 - T_0
\]
3. Arbitrary Temperature Distribution

a. Five Points:

\[
\begin{pmatrix}
V_1 d_1 \\
V_2 d_1^2 \\
V_3 d_1^3 \\
V_4 d_1^4
\end{pmatrix} =
\begin{pmatrix}
1^1 & 1^2 & 1^3 & 1^4 \\
-1 & 1 & 2 & 3 \\
(-1)^1 & (-1)^2 & (-1)^3 & (-1)^4 \\
(-2)^1 & (-2)^2 & (-2)^3 & (-2)^4
\end{pmatrix}^{-1}
\begin{pmatrix}
T_2 - T_0 \\
T_1 - T_0 \\
T_{-1} - T_0 \\
T_{-2} - T_0
\end{pmatrix}
\]

\[
\begin{pmatrix}
V_1 d_1 \\
V_2 d_1^2 \\
V_3 d_1^3 \\
V_4 d_1^4
\end{pmatrix} =
\begin{pmatrix}
-0.83333 - 0.66667 + 0.66667 + 0.83333 \\
-0.41667 - 0.66667 + 0.66667 - 0.41667 \\
0.83333 - 0.66667 + 0.66667 - 0.83333 \\
0.41667 - 0.66667 + 0.66667 + 0.41667
\end{pmatrix}
\begin{pmatrix}
T_2 - T_0 \\
T_1 - T_0 \\
T_{-1} - T_0 \\
T_{-2} - T_0
\end{pmatrix}
\]

b. Three Points:

\[
\begin{pmatrix}
V_1 d_1 \\
V_2 d_1^2
\end{pmatrix} =
\begin{pmatrix}
1^1 & 1^2 \\
(-1)^1 & (-1)^2
\end{pmatrix}^{-1}
\begin{pmatrix}
T_1 - T_0 \\
T_{-1} - T_0
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
T_1 - T_0 \\
T_{-1} - T_0
\end{pmatrix}
\]

Once the polynomial coefficients are determined, refer to Table 3.0-2 or 3.0-3 for constant \( F_1 \). Refer to Table 3.0-4 for the constants \( F_2, F_3, \) and \( F_4 \) to be used to find the fixed-end moments, reactions, and deflections of the beam.

I. Example Problem 1.

Given: The beam of rectangular cross section with a temperature distribution constant in the x-direction and varying linearly from 500°F on the top surface to 80°F on the bottom surface.
Find: Maximum stresses and maximum deflection $v(x)$ in center of beam.

Solution:

1. Find the polynomial expression representing the given temperature distribution in the $y$-direction. Refer to Paragraph 3.0.2.3, Continuous Antisymmetric Temperature, Three Points

$$v_1 d_1 = T_1 - T_0$$

Therefore

$$v_1(5) = 500 - 290$$

$$v_1 = 42$$

Therefore expression for $T = v_0 + v_1 y = 290 + 42y$.

2. Find values of $F_0, F_1, F_2,$ and $F_3$ for each term of polynomial. Refer to Table 3.0-2.

$$v_0 = 290 \quad F_0 = 290 \times b_0 d_0 = 290 \times 10(5) = 50 \times 290 \quad F_1 = 0$$

$$v_1 = 42 \quad F_0 = 0$$

$$F_1 = \frac{(42)}{12} b_0 d_0^3 = \frac{42}{12} (5) (10)^3 = 416.67(42)$$

Refer to Table 3.0-4:

$$(q = 0, \ H = 0, \ G = 0) \quad f(x) = 500 \left(\frac{x}{L}\right)^q$$

$F_2 = -1.000$ since constant in $x$-direction

$q = 0$

$F_3 = 0$ constant depth $G = 0$

constant width $H = 0$
3. Refer to Paragraph 3.0.2.2 II (Case b)

\[ M_{0z} = \alpha EF_1F_2 \]

\[ M_{0z} = 6 \times 10^{-6} \times 30 \times 10^6 \times 416.67 \times 40 \times (-1.00) \]

\[ = 3.15 \times 10^6 \]

\[ \sigma_{xx} = -\alpha ET + \frac{M_{0z} v}{l_z} \]

Top Fiber:

\[ \sigma_{xx} = -6 \times 10^6 \times 30 \times 10^6 \times 500 + \frac{(-3.15 \times 10^6) \times 5}{416.67} \]

\[ \sigma_{xx} = -127 \, 800 \text{ psi} \]

Deflection: \( v(x) = 0 \)

Therefore the beam remains straight.

II. Example Problem 2.

Given: The I-beam shown below with a linear varying temperature in the \( x \)-direction and varying as shown in the \( y \)-direction.

![Diagram of I-beam with temperature distribution](image)

Find: Stress \( \sigma_{xx} \) and deflection \( v(x) \).

Solution:

1. Find the polynomial expression representing the given temperature distribution in the \( y \)-direction. Refer to Paragraph 3.0.2.3, Arbitrary Temperature Distribution, Five Points.
\[
\begin{bmatrix}
V_1 d_1 \\
V_2 d_1^2 \\
V_3 d_1^3 \\
V_4 d_1^4
\end{bmatrix} = \begin{bmatrix}
-0.08333 & 0.6667 & -0.6667 & 0.08333 \\
-0.04167 & 0.6667 & 0.6667 & -0.04167 \\
0.08333 & -0.16667 & 0.16667 & -0.08333 \\
0.04167 & -0.16667 & -0.16667 & 0.04167
\end{bmatrix} \begin{bmatrix}
350-150 \\
225-150 \\
100-150 \\
100-150
\end{bmatrix} = \begin{bmatrix}
62.5 \\
10.42 \\
0 \\
2.083
\end{bmatrix}
\]

\[
T = 150 + \frac{62.5}{1.156} y + \frac{10.42}{(1.156)^2} y^2 + \frac{2.083}{(1.156)^4} y^4
\]

\[
T = 150 + 54.1y + 7.81y^2 + 1.168y^4
\]

2. Refer to Tables 3.0-2 and 3.0-4 for values of \( F_0 \), \( F_1 \), \( F_2 \), \( F_3 \), and \( F_4 \) for each term of polynomial.

\[
n = 0 \quad V_0 = 150 \quad F_0 = 150(1.715) = 257.3 \quad F_1 = 0
\]

\[
n = 1 \quad V_1 = 54.1 \quad F_0 = 54.1(0) = 0.0 \quad F_1 = 54.1(2.774) = 150.0
\]

\[
n = 2 \quad V_2 = 7.81 \quad F_0 = 7.81(2.774) = 21.6 \quad F_1 = 0
\]

\[
n = 4 \quad V_4 = 1.168 \quad F_0 = 1.168(6.745) = 7.9 \quad F_1 = 0
\]

\[
\Sigma F_0 = 286.8 \quad F_1 = \frac{286.8}{150.0} = 1.502
\]

\[
f \left( \frac{X}{L} \right) = \left( \frac{X}{L} \right)^q = 350 - 250 \left( \frac{X}{L} \right)
\]

\[
q = 0 \quad F_2 = -1.000 \quad F_3 = 0.0 \quad F_4 = -1.502
\]

\[
q = 1 \quad F_2 = 0.0 \quad F_3 = -1.0 \quad F_4 = -0.502
\]

\[
F_2 = -1.000 \quad F_3 = -1.0 \quad F_4 = -2.004
\]

3. Refer to Section 3.0.2.2-III (Case b).

\[
M_{0z} = -3/2 \alpha EF_1 = -3/2(180)(150) = -40,500
\]

\[
M_z = M_{0z}(1 - x/L) \quad I_z = 10.55 \text{ in}^4.
\]
\[ \sigma_{xx}(x=0) = -\alpha ET + \frac{M_{0z}y}{I_z} \]

\[ \sigma_{xx} = -180(350) - \frac{40000(2.312)}{10.55} \]

\[ \sigma_{xx} = -71860 \text{ psi} \]

Deflection:

\[ v(x) = \frac{\alpha F_1}{I_z} x^2/4(1 - x/L) = 21.35 \times 10^{-6} x^2(1 - x/L) \]

<table>
<thead>
<tr>
<th>x</th>
<th>v(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>90 \times 21.35 \times 10^{-6}</td>
</tr>
<tr>
<td>20</td>
<td>320 \times 10^{-6}</td>
</tr>
<tr>
<td>40</td>
<td>960 \times 10^{-6}</td>
</tr>
<tr>
<td>50</td>
<td>1250 \times 10^{-6}</td>
</tr>
<tr>
<td>60</td>
<td>1440 \times 10^{-6}</td>
</tr>
<tr>
<td>80</td>
<td>1280 \times 10^{-6}</td>
</tr>
<tr>
<td>90</td>
<td>810 \times 10^{-6}</td>
</tr>
<tr>
<td>100</td>
<td>0</td>
</tr>
</tbody>
</table>

Refer also to Fig. 3.0-6 for the deflection.

3.0.3 Indeterminate Beams and Rigid Frames.

Continuous beams or frames can be analyzed by the method of moment distribution described in Section B5.0, Frames. In this method a continuous beam is fixed against rotation at all the intermediate supports over which the beam is continuous; and, in the case of a frame, all the rigid joints are also fixed against rotation and displacement. For each beam segment the moment required to keep the slope at each of the two supports unchanged is called a fixed-end moment. According to the convention of the method of moment distribution, the fixed-end moment, which is an externally applied moment, is positive if it is clockwise and negative if it is counterclockwise. This convention should not be confused with the convention of strength-of-materials for internal movement (refer to Paragraph 3.0.2). Fixed-end moment is designated by the subscript, \( F \), and is abbreviated F.E.M.
The magnitude of F.E.M. is obtained by analyzing the fixed-fixed beam or the fixed-hinged beam cases of Paragraph 3.0.2. In that paragraph, $M_{0z}$ is positive when counterclockwise; so, to convert $M_{0z}$ to F.E.M., one must multiply by -1.

With this information, a normal moment distribution technique can be performed to solve any continuous beam or frame.
3.0.4 Curved Beams.

For a free curved beam of arbitrary, constant cross section, the centerline of which is an arc of a circle lying in one of the centroidal principal planes under a temperature distribution \( T(r, \theta) \), the following comments can be made.

1. For a small depth-to-radius ratio, straight-beam theory can be used instead of the curved-beam theory.

2. For linear temperature variation \( T = T_1 \frac{Z}{h} \), the results of curved-beam theory compare well with the solution obtained by exact thermoelasticity method; e.g., for a rectangular beam where the ratio of outside radius to inside radius is equal to three, the maximum stress obtained by the curved-beam theory differs from the maximum stress obtained by exact solution by only 2.4 percent. Straight-beam theory, however, gives \( \sigma_{\theta\theta} = 0 \), which is considerably erroneous.

3. For quadratic temperature distribution \( T = T_2 \frac{Z^2}{h^2} \), the difference in maximum stress obtained by curved-beam theory and exact solution is only 4.9 percent for a rectangular section where the ratio of outside to inside radius is equal to 3.0.

For solutions other than those mentioned above, see Ref. 1.

3.0.5 Rings.

A ring may be regarded as a closed, thin, curved beam, and can therefore be studied by the methods given in 3.0.4 and in Ref. 1.

3.0.6 Trusses.

3.0.6.1 Statically Determinate.

Every member of a pin-jointed determinate truss is not constrained to elongate because of thermal loading. As a result, the net axial force in each member is zero. Therefore, members of a pin-jointed determinate truss are regarded as simply supported beams under thermal loading and should be analyzed accordingly.

Deflections of joints of a truss can be obtained by the conventional "dummy-load" method described in Section B4.2.2.