SECTION B8.3

TORSION OF THIN-WALLED CLOSED SECTIONS

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B8.3.0 TORSION OF THIN-WALLED CLOSED SECTIONS

A closed section is any section where the center line of the wall forms a closed curve.

The torsional analyses of thin-walled closed sections for unrestrained and restrained torsion are included. Torsional shear stress, angle of twist, and warping stresses are determined for restrained torsion. Torsional shear stress, angle of twist, and warping deformations are considered for unrestrained torsion.

Analysis of multicell closed sections is beyond the scope of this analysis. The analysis of multicell closed sections can be found in References 11 and 13.

B8.3.1 GENERAL

I. Basic Theory

The torsional analysis of thin-walled closed sections requires that stresses and deformations be determined. The torsional shear stress (τ_t), plus warping normal stress (σ_w) for restrained torsion, should be determined at any point (P) on a thin-walled closed section at an arbitrary distance (L_x) from the origin. The angle of twist (ϕ) should be determined between an arbitrary cross section and the origin plus the warping deformation (w) at any point (P) on an arbitrary cross section for unrestrained torsion.

As was the case for solid sections (see Section B8.2.1-I), two unique coefficients exist that characterize the geometry of each cross section. These coefficients are called the torsional constant (K) and the torsional section modulus (S_t) and are functions of the dimensions of the cross section.

The torsional shear stress distribution varies along any radial line emanating from the geometric centroid of the thin-walled closed section. Since the thickness of the thin-walled section is small compared with the radius, the stress varies very little through the thickness of the cross section and is assumed to be constant through the thickness at that point.

Figure B8.3.1-1A shows a typical thin-walled cross section and Figure B8.3.1-1B shows a typical element of this cross section. Equilibrium of forces in the x direction (longitudinal) will give the following equation:

$$\tau_{L_{x1}} t_1 \Delta X = \tau_{L_{x2}} t_2 \Delta X ,$$

or, since shear stresses are equal in the longitudinal and circumferential directions,

$$\tau_{t1} t_1 = \tau_{t2} t_2 .$$

This equation indicates that the product of the torsional shear stress and the thickness at any point around the cross section is constant. This constant is called the "shear flow" (q). Therefore:

$$q = \tau_t t .$$

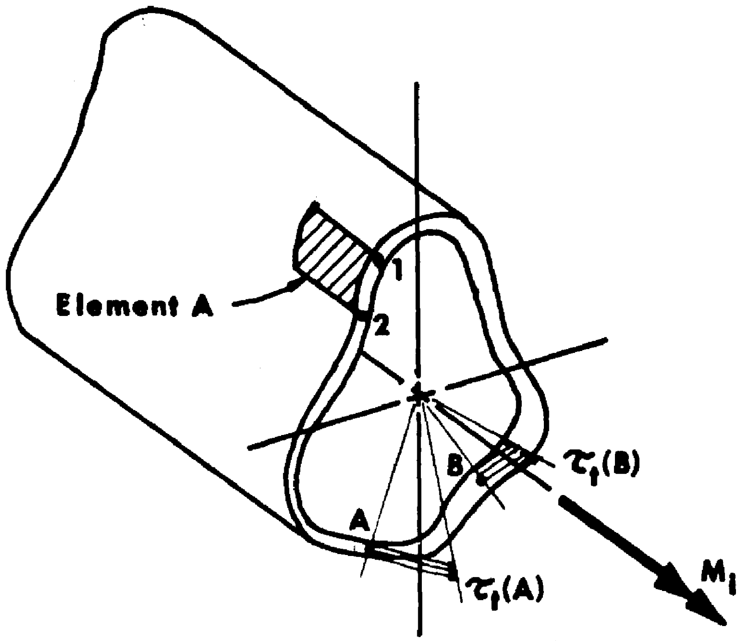
The internal forces are related to the applied twisting moment by the following equation:

$$\tau_t = \frac{M_t}{2At} = \frac{M_t}{S_t}$$

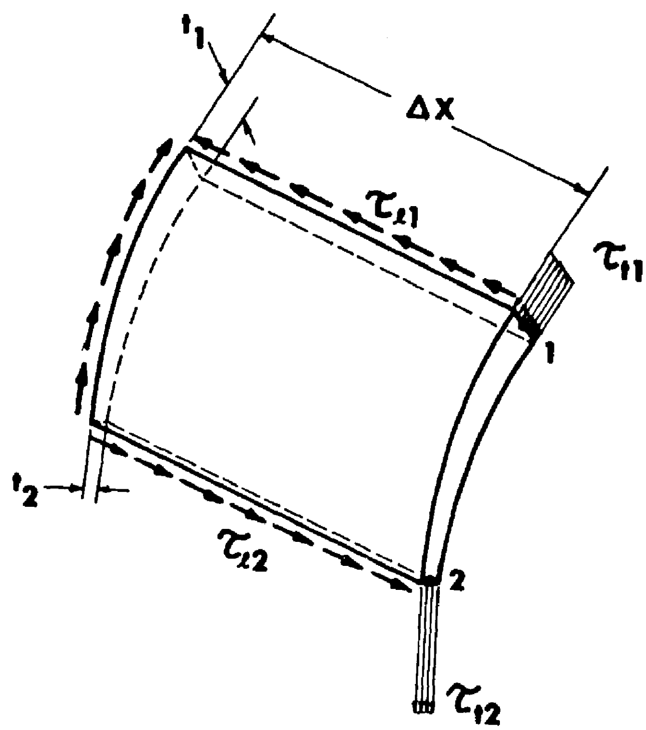
where

$$S_t = 2At$$

and A is the enclosed area of the mean periphery of the thin-walled closed section.



A. Stress Distribution and Internal Moment for Thin-Walled Cross Section



B. Stresses on Element A

FIGURE B8.3.1-1 TYPICAL THIN-WALLED CLOSED CROSS SECTION

Written in terms of shear flow, this equation becomes:

$$q = \frac{M_t}{2A} \quad .$$

II. Limitations

The torsional anslysis of thin-walled closed cross sections is subject to the following limitations:

A. The material is homogeneous and isotropic.

B. The cross section must be thin-walled, but not necessarily of constant thickness.

C. Variations in thickness must not be abrupt except at reentrant corners (see Section B8.3.3-III).

D. No buckling occurs.

E. The stresses calculated at points of constraint and at abrupt changes of applied twisting moment are not exact.

F. The applied twisting moment cannot be an impact load.

G. The bar cannot have abrupt changes in cross section.

H. The shear stress does not exceed the shearing proportional limit and is proportional to the shear strain (elastic analysis).

III. Membrane Analogy

The same use can be made of the stress function represented by the surface ABDE (Fig. B8.3.1-2) in solving the problem of the torsional resistance of a thin-walled tube as was made of the function in Section B8.2.1-III for the solid bar. These uses are as follows:

A. The twisting moment (M_t) to which the thin-walled tube is subjected is equal to twice the volume underneath the surface ABDE and is, therefore, given approximately by the equation

$$M_t = 2AH$$

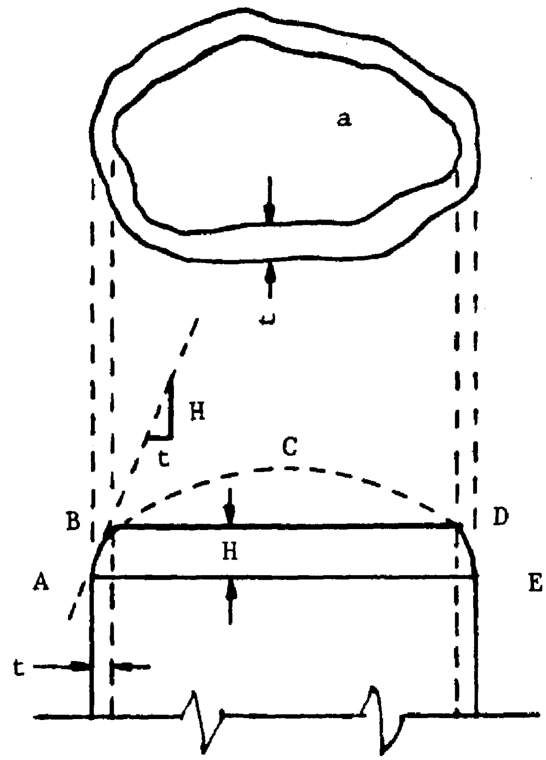


FIGURE B8.3.1-2 MEMBRANE ANALOGY FOR TORSION OF THIN-WALLED CLOSED CROSS SECTION

where A is defined as in Section B8.3.1-I and H is the height of the plane BD above the cross section.

B. The slope of the surface at any point is equal to the stress in the bar in a direction perpendicular to the direction in which the slope is taken. Hence, the slope at any point along the arcs AB or DE may be taken as H/t. The maximum shearing stress in a hollow bar at any point is, therefore,

$$\tau_t = H/t \; .$$

It can be seen that H is the same quantity as "shear flow," defined in Section B8.3.1-I.

IV. Basic Torsion Equations For Thin-Walled Closed Sections

A. Torsional Shear Stress

The basic equation for determining the torsional shear stress at an arbitrary cross section is:

$$\tau_t = \frac{M(x)}{S_t(x,s)} = \frac{M(x)}{2A(x)\,t(x,s)} = \frac{q(x)}{t(x,s)}$$

where

$$S_t(x,s) = 2A(x)\ t\ (x,s)$$

$$Q(x) = \frac{M(x)}{2A(x)}$$

and A is defined as in Section B8.3.1-I.

$M(x)$ or $q(x)$ is evaluated for $x = L_x$ at the arbitrary cross section where the torsional shear stress is to be determined, and $t(x,s)$ is evaluated at the arbitrary cross section at the point (s) on the circumference of the arbitrary cross section.

If a constant torque is applied to the end of the bar and the cross section is constant along the length of the bar, the equations reduce to:

$$\tau_t = \frac{M_t}{S_t(s)} = \frac{M_t}{2At(s)} = \frac{q}{t(s)}$$

where

$$q = \frac{M_t}{2A} \quad .$$

In the equations for torsional shear stress in Sections B8.3.2 and B8.3.3, which follow, $M(x)$ is equal to M_t and $A(x)$ is constant and equal to A. The equations in these sections determine the shear stress at any point (s) around the cross section.

B. Angle of Twist

The basic equation for determining the angle of twist between the origin and any cross section located at a distance L_x from the origin is:

$$\phi = \frac{1}{G} \int_0^{L_x} \frac{M(x)}{K(x,s)} dx = \frac{1}{4G} \int_0^{L_x} \left\{ \frac{M(x)}{A^2(x)} \left[\int_0^C \frac{ds}{t(s)} \right] (x) \right\} dx$$

where C is equal to the length of the wall center line (circumference) and

$$K(x,s) = 4A^2(x) \Big/ \int_0^C \frac{ds}{t(s)} \quad .$$

When M(x) is a constant torsional moment applied at the end of the bar, A is a constant, and t is not a function of x, the equation reduces to:

$$\phi = \frac{M_t l}{GK(s)}$$

where

$$K(s) = 4A^2 \Big/ \int_0^C \frac{ds}{t(s)} \quad .$$

When t is a constant, the equation reduces to:

$$\phi = \frac{M_t l}{GK}$$

where

$$K = \frac{4A^2 t}{C} \quad .$$

The total twist of the bar is:

$$\phi\ (\max) = \frac{M_t l}{GK}$$

where

$$K = \frac{4A^2 t}{C} \quad .$$

C. Warping Deformation

The basic equation for determining the warping deformation (w) at any point (P) on an arbitrary cross section located at a distance $x = L_x$ from the origin is:

$$w(s) - w_o = \frac{M_t}{2AG} \int_0^s \left[\frac{1}{t(s)} - r(s) \left(\frac{\int_0^C \frac{ds}{t(s)}}{2A} \right) \right] ds$$

where w(s) is the warping deformation at point (s) measured from the x-y plane through the origin of the s coordinate system; w_o is the distance from the x-y plane through the origin of the mean displacement plane; and r(s) is the normal distance to a line parallel to the increment of arc length ds (see Example Problem 2, Section B8.3.4-II).

The mean displacement plane, which is located at the same z coordinate as the undeformed cross section, will pass through those points on the cross section that lie on axes of symmetry (see Example Problem 2, Section B8.3.4-II). For unsymmetrical sections, the point (s), measured from an assumed arc length origin through which the mean displacement plane passes, is determined by evaluating the following integral for s.

$$\int_0^s \left[\frac{1}{t(s)} - r(s) \left(\frac{\int_0^C \frac{ds}{t(s)}}{2A} \right) \right] ds = 0 \quad .$$

D. Warping Stresses

Warping stress calculations are very complicated and cannot be put into a generalized form. Techniques for evaluating these stresses can be found in Reference 1.

Warping stresses for a rectangular section are included in section B8.3.3-II.

B8.3.2 CIRCULAR SECTIONS

I. Constant-Thickness Circular Sections

A constant-thickness, circular, thin-walled closed section experiences no warping for unrestrained torsion and develops no warping normal stresses for restrained torsion.

The torsional shear stress is determined by the following equation:

$$\tau_t = \frac{M_t}{S_t}$$

where

$$S_t = 2At \ .$$

The torsional shear stress defined in terms of shear flow is determined by the following equation:

$$\tau_t = q/t$$

where

$$q = \frac{M_t}{2A} \ .$$

The total angle of twist is determined by the following equation:

$$\phi\,(\max) = \frac{M_t L}{GK}$$

where

$$K = \frac{4A^2 t}{C} \ .$$

II. Varying Thickness Circular Sections

A circular thin-walled closed section with varying thickness will warp for unrestrained torsion, and warping normal stresses are developed for restrained torsion. The warping normal stresses and warping deformations are negligible and can be neglected when the change in thickness is small and gradual.

The torsion shear stress is calculated in the same manner as for constant-thickness circular sections, except that t is now a function of s.

The total angle of twist for a circular thin-walled closed section with varying thickness is determined by the following equation:

$$\phi\,(\max) = \frac{M_t L}{GK}$$

where

$$K = \frac{4A^2}{\int_0^C \frac{ds}{t(s)}} \quad .$$

B8.3.3 NONCIRCULAR SECTIONS

I. Unrestrained Torsion

Noncircular sections experience warping for unrestrained torsion, except for the case noted below, and develop no warping normal stresses.

Note that no warping occurs in a cross section that has a constant value for the product rt around the circumference of the cross section.

Longitudinal warping deformations are usually not of concern and are not evaluated. The use of the basic equation for determining warping deformations for closed sections (see Section B8.3.1-IVC) is used in Example Problem II (see Section B8.3.4-III).

A. Elliptical Section

The torsional shear stress for constant thickness is determined by the following equation:

$$\tau_t = \frac{M_t}{S_t}$$

where

$$S_t = 2At$$

and

$$A = \pi \left[ab - \frac{t}{2}\,(a + b) + \frac{t^2}{4}\right] \quad .$$

The values of a, b, and t are defined in Figure B8.3.3.-1.

The torsional shear stress defined in terms of shear flow is determined by the following equation:

$$\tau_t = \frac{q}{t}$$

where

$$q = \frac{M_t}{2A}$$

and A is defined as above.

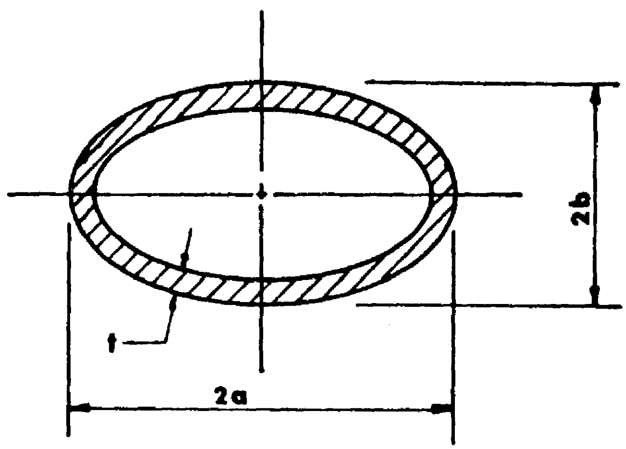


FIGURE B8.3.3-1 ELLIPTICAL SECTION

The torsional shear stress for varying thickness is calculated in the same manner as constant thickness, except that it is now a function of s, t(s).

The total angle of twist for constant thickness is determined by the following equation:

$$\phi\ (\max) = \frac{M_t L}{GK}$$

where

$$K = \frac{4A^2 t}{C}$$

A is defined above, and the equation for C is, approximately,

$$C = \pi(a + b - t)\left[1 + 0.27\,\frac{(a - b)^2}{(a + b)^2}\right] .$$

The total angle of twist for varying thickness is determined by the following equation:

$$\phi\ (\max) = \frac{M_t L}{GK}$$

where

$$K = \frac{4A^2}{\int_0^C \frac{ds}{t(s)}}$$

and the area is as defined in Section B8.3.1-I.

B. <u>Rectangular Section (Constant Thickness)</u>

The torsional shear stress for constant thickness is determined at points A and B (Fig. B8.3.3-2) by the following equation:

$$\tau_t = \frac{M_t}{S_t}$$

where

$$S_t = 2At$$

and $$A = ab - t(a+b) + t^2 .$$

The values of a, b, and t are defined in Figure B8.3.3-2.

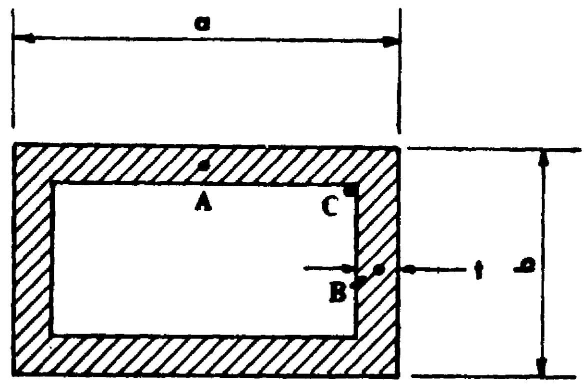


FIGURE B8.3.3-2 RECTANGULAR SECTION (CONSTANT THICKNESS)

The torsional shear stress defined in terms of shear flow is determined by the following equation:

$$\tau_t = \frac{q}{t}$$

where

$$q = \frac{M_t}{2A}$$

and A is defined as above.

The stresses at the inner corners (points C on Fig. B8.3.3-2) will be higher than the stresses calculated at points A and B unless the ratio of the radius of the fillet to thickness is greater than 1.5. For small radius rectangular section stresses see Section B8.3.3-III.

The total angle of twist for constant thickness is determined by the following equation:

$$\phi(\max) = \frac{M_t L}{GK}$$

where

$$K = \frac{4A^2 t}{C} \quad .$$

A is defined above, and $C = 2(a + b - 2t)$.

C. Rectangular Sections (Different Thickness)

The torsional shear stress for different but nonvarying thickness is determined at points A and B (Fig. B8.3.3-3) by the following equation:

$$\tau_t = \frac{M_t}{S_t}$$

where

$$S_t = 2At_1$$

for point A,

$$S_t = 2At_2$$

for point B, and

$$A = (a - t_2)(b - t_1) \quad .$$

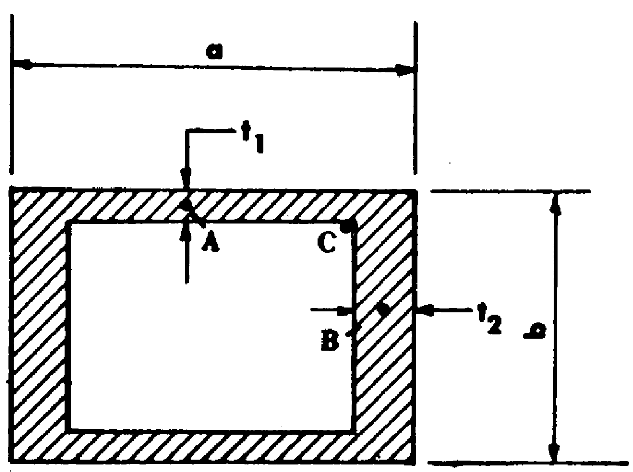


FIGURE B8.3.3-3 RECTANGULAR SECTION (DIFFERENT THICKNESSES)

The torsional shear stress defined in terms of shear flow is determined by the following equation:

$$\tau_t = \frac{q}{t_1}$$

for point A,

$$\tau_t = \frac{q}{t_2}$$

for point B, and

$$q = \frac{M_t}{2A}$$

where A is defined as above.

The stress at the inner corners (points C) will be higher than the stresses calculated at points A and B unless the ratio of the radius of the fillet to the thickness is greater than 1.5. For small radius rectangular section stresses see Section B8.3.3-III.

The total angle of twist for different but nonvarying thickness is determined by the following equation:

$$\phi\ (max) = \frac{M_t L}{GK}$$

where

$$K = \frac{2t_1t_2(a - t_2)^2\ (b - t_1)^2}{at_1 + bt_2 - t_1^2 - t_2^2} .$$

D. <u>Arbitrary Section (Constant Thickness)</u>

The torsional shear stress for an arbitrary section with constant thickness (Fig. B8.3.3-4) is determined by the following equation:

$$\tau_t = \frac{M_t}{S_t}$$

where

$$S_t = 2At$$

and A is defined in Section B8.3.1-I.

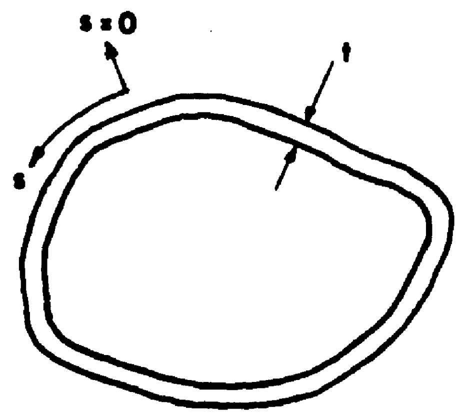


FIGURE B8.3.3-4 ARBITRARY SECTION (CONSTANT THICKNESS)

The torsional shear stress defined in terms of shear flow is determined by the following equation:

$$\tau_t = \frac{q}{t}$$

where

$$q = \frac{M_t}{2A}$$

and A is defined as above.

The total angle of twist is determined by the following equation:

$$\phi\ (\max) = \frac{M_t L}{GK}$$

where

$$K = \frac{4A^2 t}{C} .$$

A is defined as above, and C, the circumference, is defined as follows:

$$C = \int_0^C ds \quad .$$

E. Arbitrary Section (Varying Thickness)

The torsional shear stress for an arbitrary section with varying thickness (Fig. B8.3.3-5) is determined by the following equation:

$$\tau_t = \frac{M_t}{S_t}$$

where

$$S_t = 2At$$

and A is defined in Section B8.3.1-I. The shear flow is determined by the following equation:,

$$\tau_t = \frac{q}{t}$$

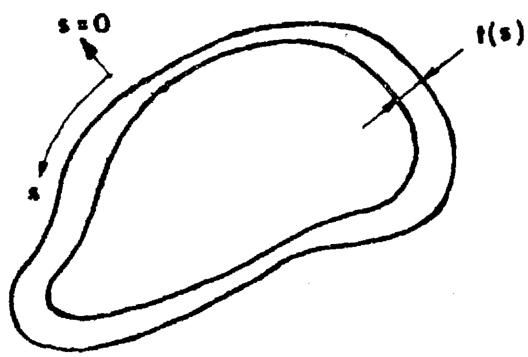


FIGURE B8.3.3-5 ARBITRARY SECTION (VARYING THICKNESS)

where

$$q = \frac{M_t}{2A}$$

and A is defined as above.

Note that the maximum shear stress occurs at the point of least thickness. The total angle of twist is determined by the following equation:

$$\phi(\max) = M_t \frac{L}{GK}$$

where

$$K = 4A^2 / \int_0^C \frac{ds}{t(s)}$$

and A is defined as above.

II. Restrained Torsion

Restrained torsion of noncircular closed sections occurs at fixed ends and at points of abrupt change in torque.

The warping normal stresses associated with the restrained torsion attenuate rapidly, and their analytical determination is extremely difficult.

Torsional shear stresses associated with these restraints are calculated as in Section B8.3.3-I. The warping normal stress for the rectangular section shown in Figure B8.3.3-2 is determined by the following equation:

$$\sigma_w(\max) = \frac{m}{K}$$

where

$$K = \frac{tG}{4M_t}(a+b)^2\left(\frac{1-\mu^2}{E}\right)\left(\frac{2}{1-\mu}\right)^{\frac{1}{2}}$$

and m is obtained from Figure B8.3.3-6.

III. Stress Concentration Factors

The curve in Figure B8.3.3-7 gives the ratio of the stress at the reentrant corners to the stress along the straight sections at points A and B shown in Figure B8.3.3-2.

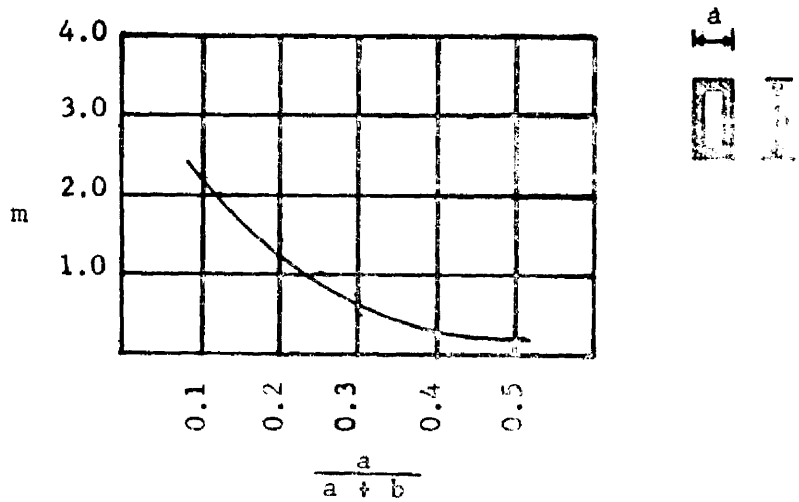


FIGURE B8.3.3-6 VALUE OF m FOR RECTANGULAR SECTION

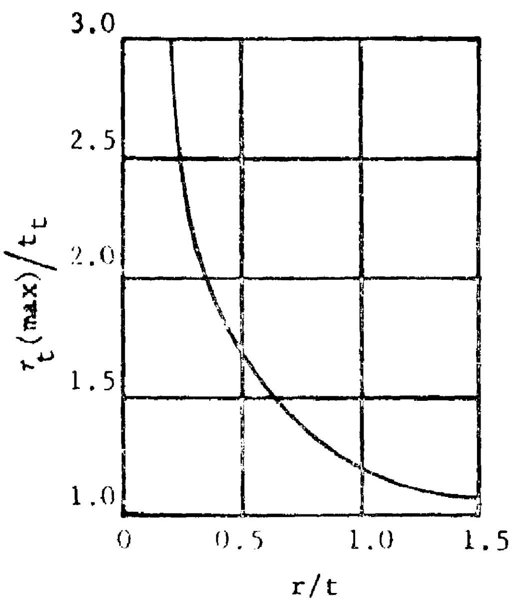


FIGURE B8.3.3-7 STRESS CONCENTRATION FACTORS AT REENTRANT CORNERS

B8.3.4 EXAMPLE PROBLEMS FOR TORSION OF THIN-WALLED CLOSED SECTIONS

I. Example Problem 1

For the problem shown in Figure B8.3.4-1, it is required to find the following:

A. Shear stresses at points A, B, and C on the cross section and maximum angle of twist caused by the torsional load.

B. Local normal stresses caused by restraint at the fixed end.

Solution:

A. From Section B8.3.3-IB, the shear stress at points A and B is

$$\tau_t = \frac{M_t}{2At}$$

where $A = ab - t\,(a + b) + t^2$.

Therefore, $A = 6(3) - 0.2\,(6 + 3) + (0.2)^2$

$$A = 16.24 \text{ in.}^2$$

and $$\tau_t = \frac{100,000}{2(16.24)\,(0.2)}$$

$$\tau_t = 15,394 \text{ psi} \quad .$$

For the maximum shear stress at point C, refer to Figure B8.3.3-7 for the ratio of the stress at point C to the stress at points A and B.

$$\frac{\tau_t(\max)}{\tau_t(A)} = 1.7$$

$$\tau_t(\max) = 1.7\,(15,394)$$

$$\tau_t(\max) = 26,170 \text{ psi}$$

The maximum angle of twist is determined by the following equation from Section B8.3.3-IB:

$$\phi(\max) = \frac{M_t L}{GK}$$

where

$$K = \frac{4A^2 t}{C}$$

and $C = 2(a + b - 2t)$.

Therefore, $C = 2(6 + 3 - 2\ (0.2))$

$$C = 17.2$$

$$K = \frac{4\ (16.24)^2 0.2}{17.2}$$

$$K = 12.27$$

$$\phi(\max) = \frac{100,000\ (60)}{4 \times 10^6\ (12.27)}$$

$$\phi(\max) = 0.122 \text{ radians} .$$

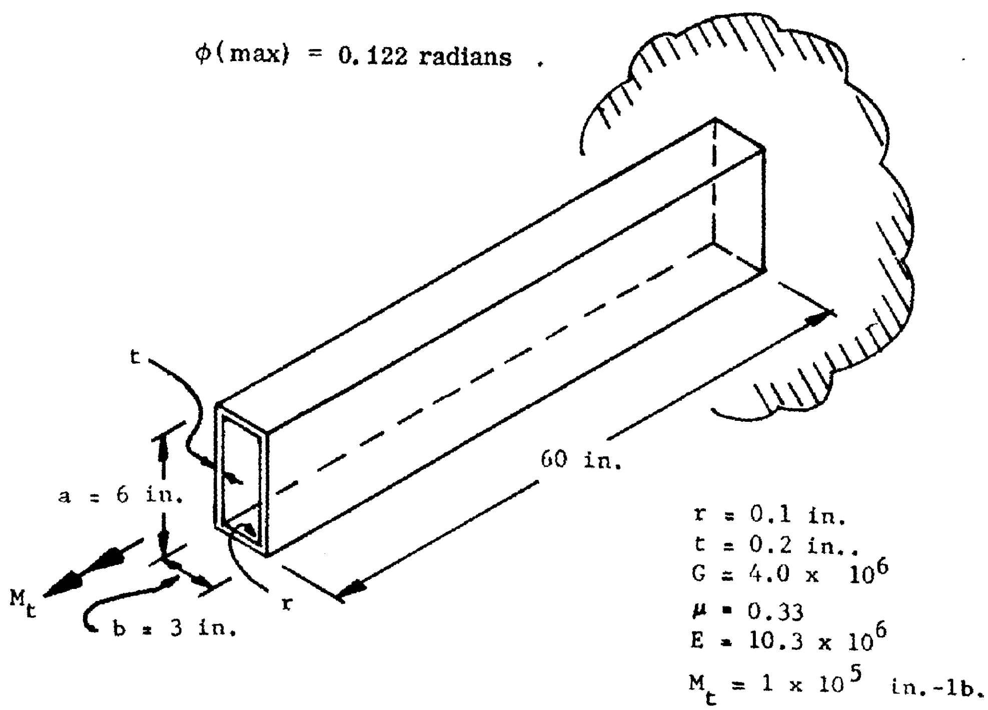


FIGURE B8.3.4-1

B. To find the normal stress at the fixed end, use the equations in Section B8.3.3-II.

From Figure B8.3.3-6,

$$m = 0.3$$

$$K = \frac{tG}{4M_t} (a + b)^2 \left(\frac{1 - \mu^2}{E}\right) \left(\frac{2}{1 - \mu}\right)^{\frac{1}{2}}$$

$$K = \frac{(0.2)(4 \times 10^6)}{4 \times 10^5} (3 + 6)^2 \sqrt{\frac{2}{1 - 0.33}} \left(\frac{1 - 0.33^2}{10.3 \times 10^6}\right)$$

$$K = 24.217 \times 10^{-6}$$

$$\sigma = \frac{0.33}{24.217 \times 10^{-6}} = 13,620 \text{ psi}$$

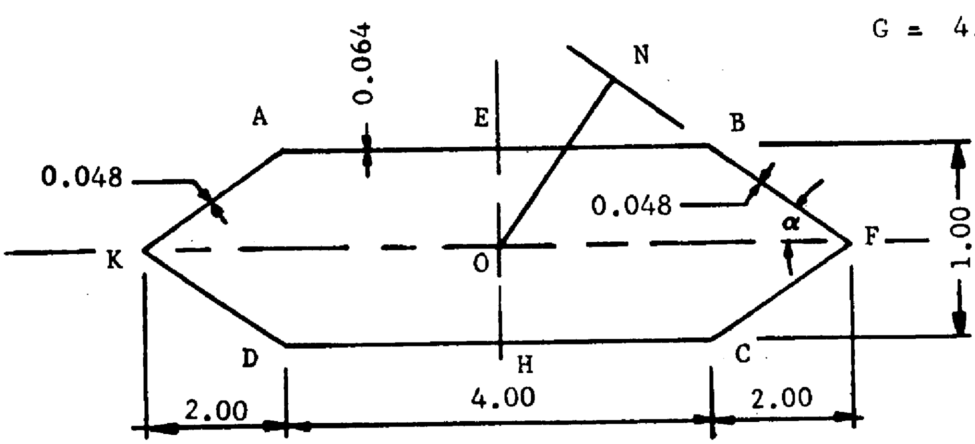
II. Example Problem 2

For the cross section shown in Figure B8.3.4-2, it is required to find the warping deformations of the points shown.

(dimensions are in inches)

$M_t = 1 \times 10^5$ in.-lb.

$G = 4.0 \times 10^6$



Determine the distribution of warping

$\sin \alpha = 0.2425$ ∴ ON = 0.97

FIGURE B8.3-4-2

From Section B8.3.1-IVC,

$$w(s) - w_o = \frac{M_t}{2AG} \int_0^s \left[\frac{1}{t(s)} - r(s) \left(\frac{\int_0^C \frac{ds}{t(s)}}{2A} \right) \right] ds$$

or

$$\frac{G}{M_t} [w(s) - w_o] = \frac{1}{2A} \int_0^s \left[\frac{1}{t(s)} - r(s) \left(\frac{\int_0^C \frac{ds}{t(s)}}{2A} \right) \right] ds \quad .$$

A = enclosed area of section = 6.0 in.^{2}

$$\int_0^C \frac{ds}{t(s)} = 2 \left\{ \frac{KD}{0.048} + \frac{DC}{0.064} + \frac{CF}{0.048} \right\} = 2 \left\{ \frac{2.062}{0.048} + \frac{4}{0.064} + \frac{2.062}{0.048} \right\}$$

$$= 296.8 \quad .$$

Choose point D as the origin and measure s from point D. For sector DC:
$r = 0.5 \quad t = 0.064$

$$\frac{2AG}{M_t} \left[w_C - w_D \right] = \int_0^4 \left[\frac{1}{0.064} - \frac{0.5(296.8)}{2(6)} \right] ds$$

$$= 3.26(4)$$

$$= \underline{13.04}$$

For sector CF: $r = 0.97 \quad t = 0.048$

$$\frac{2AG}{M_t} \left[w_F - w_C \right] = \int_0^{2.062} \left[\frac{1}{0.048} - \frac{0.97(296.8)}{2(6)} \right] ds$$

$$= -(3.16)(2.062)$$

$$= -6.52 \quad .$$

For sector FB: r = 0.97 t = 0.048 (Same as sector CF)

$$\frac{2AG}{M_t}\left[w_B - w_F\right] = -\underline{6.52}$$

For sector BA: (same as sector DC)

$$\frac{2AG}{M_t}\left[w_A - w_B\right] = \underline{13.04}\ .$$

For sector AK: (same as sector CF)

$$\frac{2AG}{M_t}\ w_K - w_A = -\underline{6.52}\ .$$

For sector KD: (same as sector CF)

$$\frac{2AG}{M_t}\left[w_D - w_K\right] = -\underline{6.52}\ .$$

Hence, the summation of warping deflections around the section from D back to D equals zero.

Now it is desired to find the mean displacement plane (see Fig. B8.3.4-3). From symmetry, points E_1, F_1, H_1, and K_1 will lie on the mean displacement plane and will not deform from their original positions. Therefore the distance from point D to the mean displacement plane is

$$\frac{2AG}{M_t}\left[w_D - w_0\right] = \underline{6.52} \quad .$$

Therefore the warping deformations are

$$w_D = w_C = w_B = w_A = \frac{6.52\ M_t}{2AG}$$

$$= \frac{6.52(1 \times 10^5)}{2(6)(4 \times 10^6)} = 0.0136 \text{ inch}$$

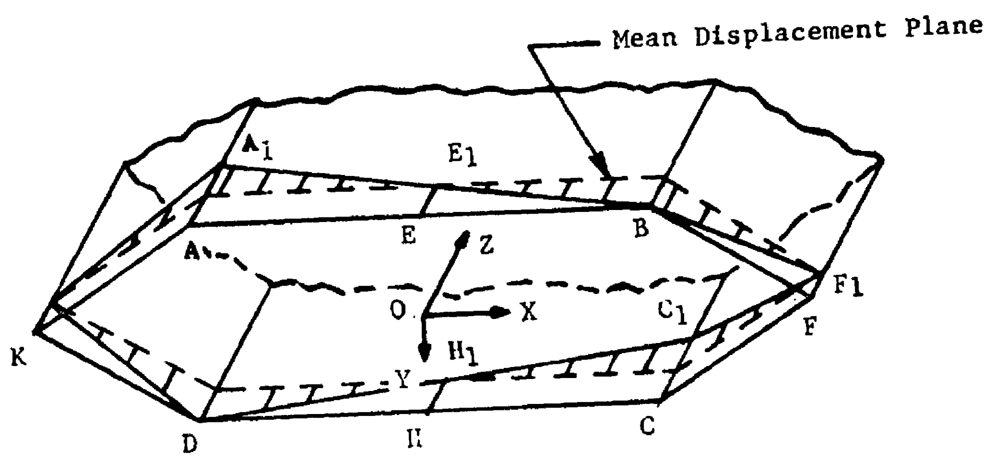


FIGURE B8.3.4-3

III. Example Problem 3

For the problem shown in Figure B8.3.4-4, it is required to find the following:

A. Maximum torsional shear stress at $L_x = 0$, $L_x = \frac{L}{2}$, and $L_x = L$.

B. Maximum angle of twist.

Solution:

A. The formula for shear stress as obtained by Section B8.3.1-IVA is

$$\tau_t = \frac{M(x)}{2A(x)\ t(x,s)} \quad .$$

Therefore, at $L_x = 0$, since $M(x) = 0$, $\tau_t = 0$ at $L_x = \frac{L}{2}$, $M(x) = \frac{M_0}{2}$

$$A(x) = \pi r_0^2 (1 + \frac{1}{2})^2$$
$$= \pi r_0^2 \frac{9}{4} \quad .$$

The shear stress will be maximum at thickness of t_0. Therefore

$$\tau(\max) = \frac{4M_0}{2(2)\pi r_0^2\ 9\ t_0}$$
$$= \frac{M_0}{9\pi r_0^2 t_0} = \frac{100,000}{9\pi(5)^2\ 0.1} = 1,414 \text{ psi} \quad .$$

Where $L_x = L$

$$M(x) = M_0$$
$$A(x) = 4\pi r_0^2$$
$$\tau(\max) = \frac{M_0}{2(4)\pi r_0^2\ t_0} = 1,591 \text{ psi} \quad .$$

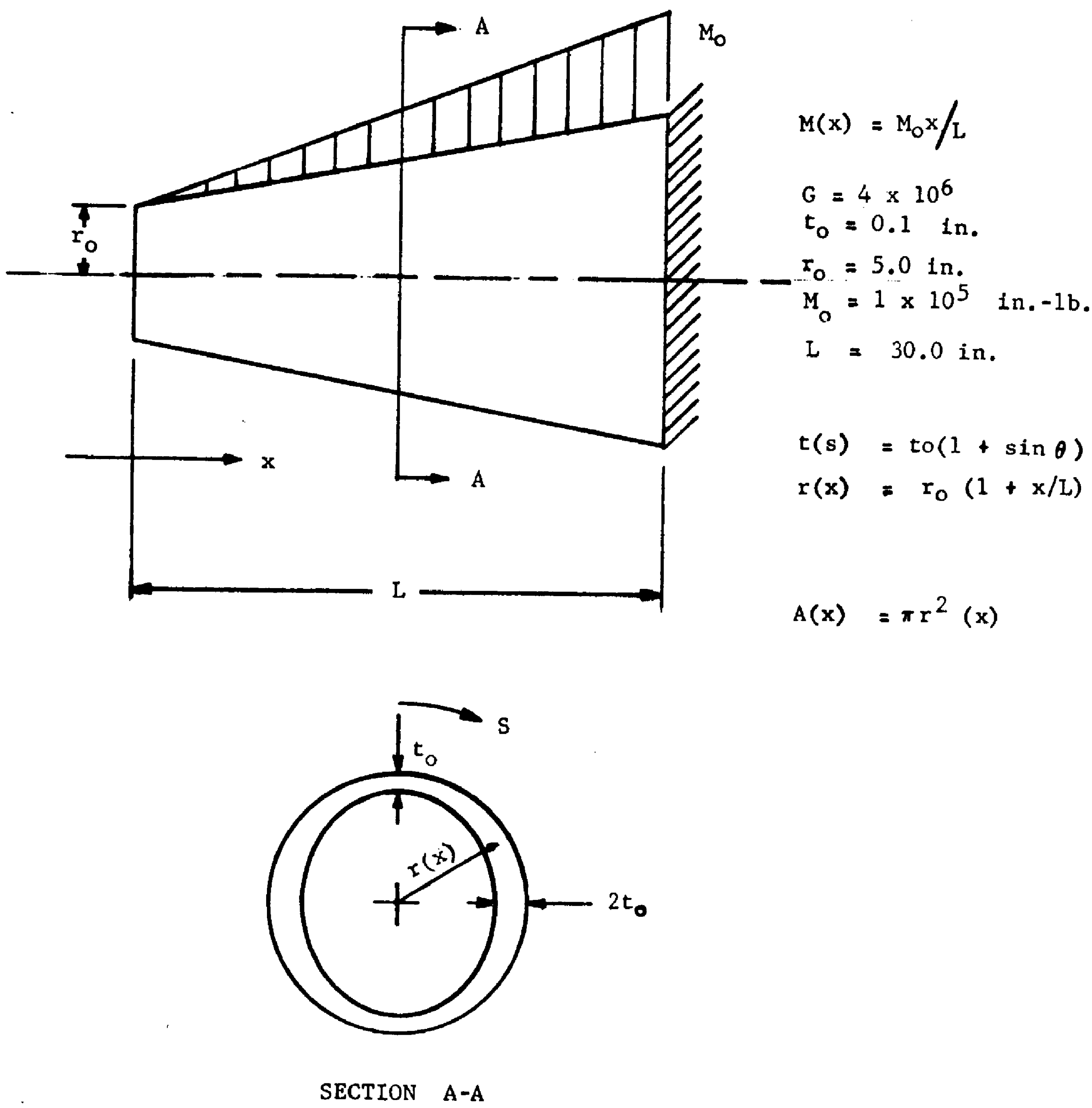


FIGURE B8.3.4-4

B. The formula for angle of twist is obtained from Section B8.3.1-IVB.

$$\phi = \frac{1}{G} \int_0^{L_x} \left\{ \frac{M(x)}{A^2(x)} \left[\int_0^C \frac{ds}{t(s)} \right](x) \right\} dx$$

$$\left[\int_0^C \frac{ds}{t(s)} \right](x) = 4 \int_0^{90} \frac{r(x)\ d\theta}{t_0(1 + \sin\theta)}$$

$$= \frac{4r(x)}{t_0} \left[-\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \right]_0^{90}$$

$$= \frac{4r(x)}{t_0}$$

Therefore, for maximum angle of twist

$$4G\phi = \frac{4}{t_0} \int_0^L \frac{M_0\left(\frac{x}{L}\right) r_0 \left(1 + \left(\frac{x}{L}\right)\right.}{\pi^2 r_0^4 \left(1 + \frac{x}{4}\right)^4} dx$$

$$G\phi = \frac{M_0}{\pi^2 r_0^3 t_0 L} \int_0^L \frac{x\ dx}{\left(1 + \frac{x}{L}\right)^3}$$

$$G\phi = \frac{M_0 L^2}{\pi^2 r_0^3 t_0 L} \left[\frac{-1}{\left(1 + \frac{x}{L}\right)} + \frac{1}{2\left(1 + \frac{x}{L}\right)^2} \right]_0^L$$

$$G\phi = \frac{M_0 L}{\pi^2 r_0^3 t_0} \left(\frac{1}{8}\right)$$

$$\phi = \frac{M_0 L}{8\pi^2 r_0^3 t_0 G} = \frac{1 \times 10^5 \times 30}{8\pi^2 (5)^3 (0.1) 4 \times 10^6}$$

$$\phi\ (\max) = 0.76 \times 10^{-3} \text{ radians}$$