SECTION B8

TORSION

B8.0.0 TORSION

Sections under B8 deal with the torsional analysis of straight structural elements that have longitudinal dimensions much greater than their cross-sectional dimensions. Such an element is called a bar.

The first division, Section B8.1, provides a common ground for the analytical divisions which follow: the solid cross section, treated in Section B8.2; the thin-walled closed cross section treated in Section B8.3; and the thin-walled open cross section, treated in Section B8.4.

In each of the divisions, the cross section under consideration will be defined, described, and pictorially represented. Particular conditions which are pertinent to the approach, such as restraints, will be stated; the basic theory, and limitations, if any, will be given.

SECTION B8.1

GENERAL

TABLE OF CONTENTS

B8. 1. 0 GENERAL

Section B8. 1 presents the notation and sign convention for local coordinate systems, applied twisting moments, internal resisting moments, stresses, deformations, and derivations of angle of twist. These conventions will be followed in Sections B8. 2, B8. 3, and B8. 4.

Restrained torsion and unrestrained torsion are considered for the thin-walled open and thin-walled closed cross sections, and unrestrained torsion is considered for the solid cross section. Restrained torsion requires that no relative longitudinal displacement shall occur between two similar points on any two similar cross sections. Warping is restrained.

Restrained torsion of solid cross sections is not considered because it is a localized stress condition and attenuates rapidly. The stresses and deformations determined by the methods contained in this section can be superimposed upon stresses and deformations caused by other types of loading if the deformations are small and the maximum combined stress does not exceed the yield stress of the material.

B8.1.1 NOTATION

All general terms used in this section are defined herein. Special terms are defined in the text as they occur.

a	Width of rectangular section, in.
A	Enclosed area of mean periphery of thin-walled closed section, $in.^2$
b	Length of element, width of flange, in.
b'	Width of flange minus thickness of web, in.
C	Length of wall centerline (circumference), in.
d	Total section depth, in.
D	Diameter of circular bar, in.
E	Young's modulus, $lb/in.^2$
G	Shear modulus of elasticity, $lb/in.^2$
h	Distance between flange centerlines, in.
I	Moment of inertia, $in.^4$
J	Polar moment of inertia, $in.^4$
K	Torsional constant, $in.^4$
L	Length of bar, in.
L_x	Arbitrary distance along x-axis from origin, in.
m_t	Applied uniform twisting moment or maximum value of varying applied twisting moment, in.-lb/in.
M_i	Internal twisting moment, in.-lb.
M_t	Applied concentrated twisting moment, in.-lb.
$M_{(x)}$	Internal twisting moment at point x along bar, written as function of x
p	Pressure, $lb/in.^2$

P	Arbitrary point on cross section
q	Shear flow, lb/in.
r	Radius of circular cross section, in.
R	Radius of circular fillet, in.
s	Distance measured along thin-walled section from origin, in.
S_t	Torsional modulus, $in.^3$
$S_w(s)$	Warping statical moment, $in.^4$
t	Thickness of element, in.
t_w	Thickness of web, in.
T	Tensile force per unit length, $lb/in.^2$
u	Displacement in the x direction, in.
v	Displacement in the y direction, in.
w	Displacement in the z direction, in.
$W_n(s)$	Normalized warping function, $in.^2$
V	Volume, $in.^3$
α	Defined in Section B8.4.1-IV
β	Defined in Section B8.4.1-IV
Γ	Warping constant, $in.^6$
γ	Shear strain
θ	Unit twist, rad/in. ($\theta = d\phi/dx = \phi'$)
μ	Poisson's ratio
ρ	Radial distance from the centroid of the cross section to arbitrary point P, in.
ρ_o	Radial distance to tangent line of arbitrary point P from shear center, in.

σ_x	Longitudinal normal stress, lb/in.^{2}
τ	Total shear stress, lb/in.^{2}
τ_t	Torsional shear stress, lb/in.^{2}
τ_ℓ	Longitudinal shear stress, lb/in.^{2}
τ_w	Warping shear stress, lb/in.^{2}
ϕ	Angle of twist, rad ($\phi = \int_0^{L_x} \theta dx$)
ϕ', ϕ'', ϕ'''	First, second, and third derivatives of angle of twist with respect to x, respectively
Φ	Saint-Venant stress function

Subscripts:

i	inside
l	longitudinal
n	normal
o	outside
s	point s
t	torsional or transverse
w	warping
x	longitudinal direction

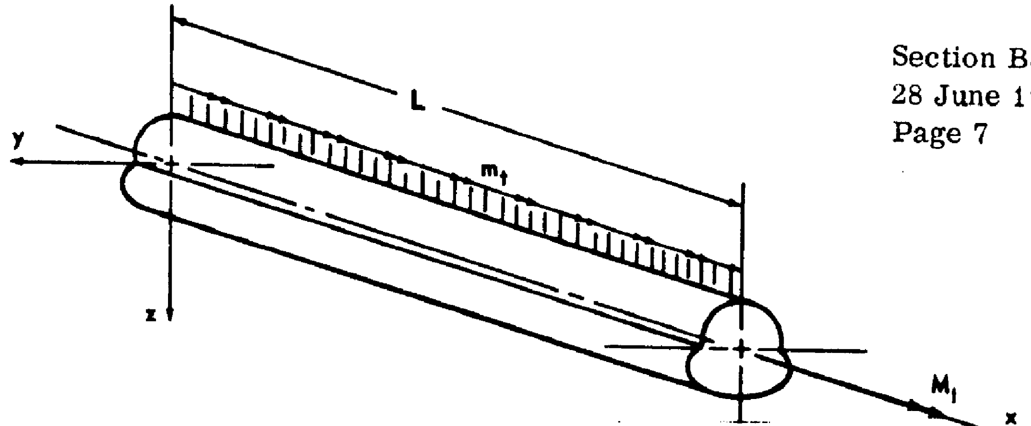
B8.1.2 SIGN CONVENTION

The local coordinate system for a bar subjected to an applied twisting moment and the sign conventions for applied twisting moments, internal resisting moments, stresses, displacements, and derivatives of displacements are defined so that there is continuity throughout the equations presented in this section.

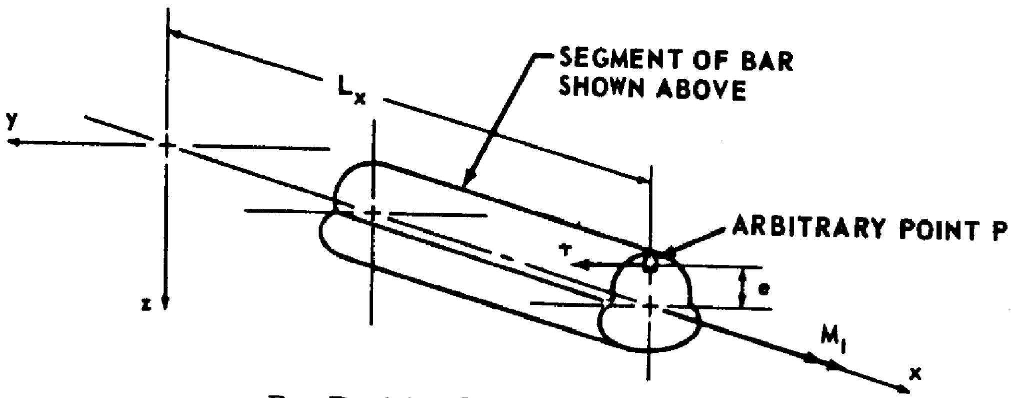
B8.1.2 SIGN CONVENTION

I. LOCAL COORDINATE SYSTEM

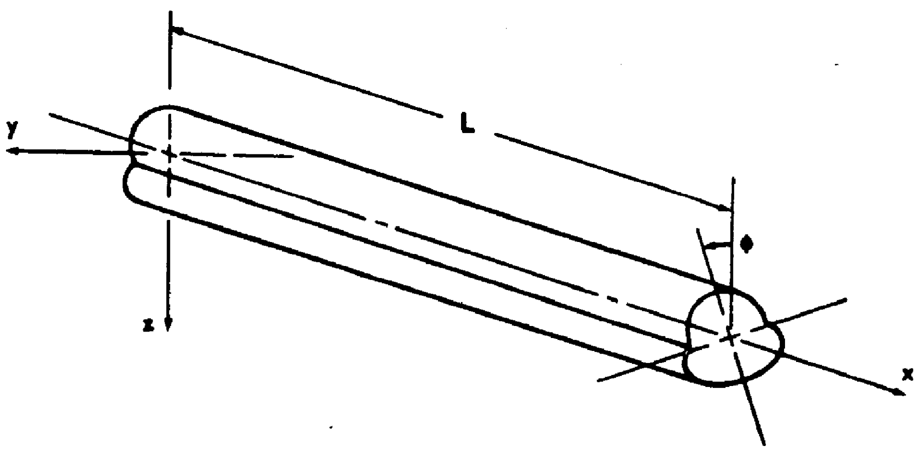
The local coordinate system is applied to either end of a bar unless specific limitations are stated. The x axis is placed along the length of the bar. The y and z axes are the axes of maximum inertia when the cross section is unsymmetrical, as may be seen in the solid cross section shown in Figure B8.1.2-1. The coordinate system and sign convention shown apply to thin-walled open and thin-walled closed cross sections also.



A. Local Coordinate System and Positive Applied Twisting Moments



B. Positive Internal Resisting Moment and Shear Stresses



C. Positive Angle of Twist

Figure B8.1.2-1. Local Coordinate System and Positive Sign Convention

B8.1.2 SIGN CONVENTION

II. APPLIED TWISTING MOMENTS

The applied twisting moments (m_t or M_t) are twisting moments about the x axis. The applied twisting moments are positive if they are clockwise when viewed from the origin or are in the positive x direction when represented vectorially. (See Fig. B8.1.2-1A.)

B8.1.2 SIGN CONVENTION

III. INTERNAL RESISTING MOMENTS

The internal resisting moments (M_i) are about the x axis and have the same sign convention as the applied twisting moment when they are evaluated on the y-z plane of a bar segment that is farthest from the origin. (See Fig. B8.1.2-1B.)

B8.1.2 SIGN CONVENTION

IV. STRESSES

Tensile normal stresses (σ_x) are positive, and compressive normal stresses are negative. Shear stresses (τ) are positive when they are equivalent to positive internal resisting moments. (See Fig. B8.1.2-1B.)

B8.1.2 SIGN CONVENTION

V. DEFORMATIONS

An applied twisting moment induces a rotation or angle of twist (ϕ) about the x axis. The rotation is positive if it is clockwise when viewed from the origin. (See Fig. B8.1.2-1C.) An applied twisting moment also induces a longitudinal displacement (u) in the x direction for unrestrained torsion. (See Fig. B8.2.2-2B.) The longitudinal displacement is positive when in the direction of the positive x axis.

B8.1.2 SIGN CONVENTION

VI. DERIVATIVES OF ANGLE OF TWIST

The first (ϕ'), second (ϕ''), and third (ϕ''') derivatives of the angle of twist with respect to the positive x coordinate are positive, positive, and negative, respectively, when the rotation is positive and a concentrated applied twisting moment (M_t) is applied at the ends of the bar.

SECTION B8.2

TORSION OF SOLID SECTIONS

TABLE OF CONTENTS

B8.2.0 TORSION OF SOLID SECTIONS

The torsional analysis of solid sections is restricted to unrestrained torsion and does not consider warping deformations.

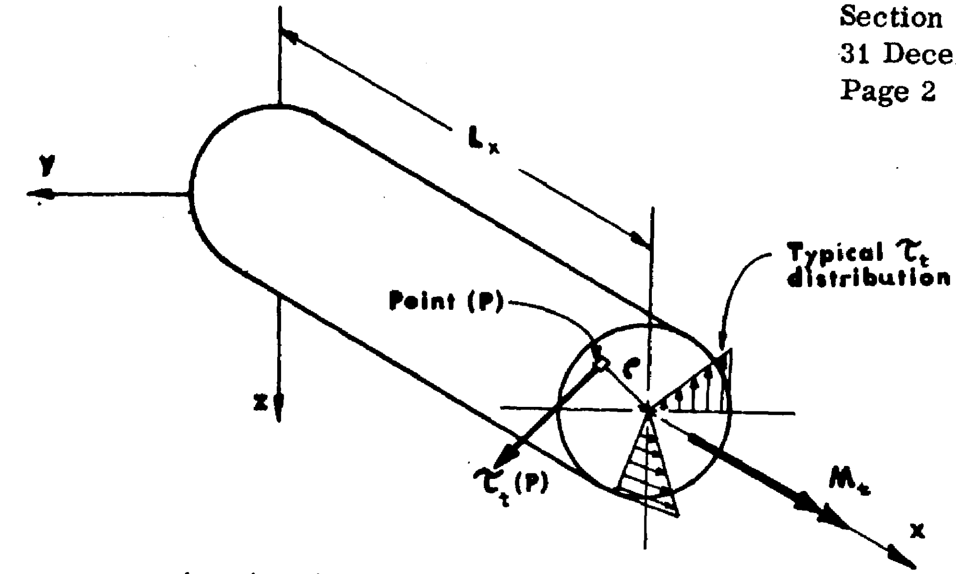
B8.2.1 General

I Basic Theory

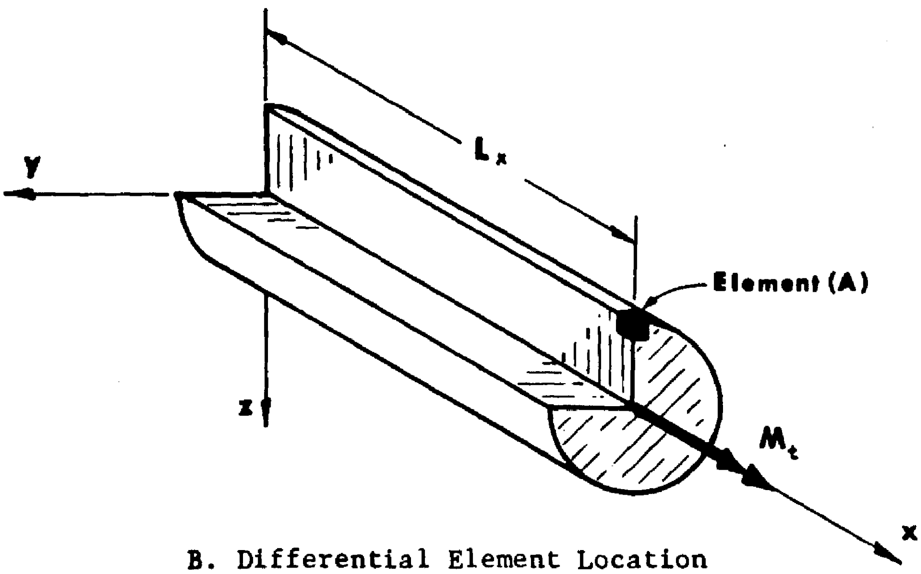
The torsional analysis of solid sections requires that stresses and deformations be determined. The torsional shear stress (τ_t) is determined at any point (P) on a cross section at an arbitrary distance (L_x) from the origin. The resulting angle of twist (ϕ) is determined between an arbitrary cross section and the origin. These shear stresses and the resulting angles of twist can be determined when the material properties of the bar, geometry of the bar, and the applied twisting moment are known.

Two unique coefficients characterize the geometry of each cross section, the torsional constant (K) and the torsional section modulus (S_t). These coefficients are functions of the dimensions of the cross section. These constants are used for calculating deformations and stresses, respectively. For a circular section, the torsional constant reduces to the polar moment of inertia (J), and the torsional section modulus reduces to J/ρ; but for all other cross sections they are more complex functions.

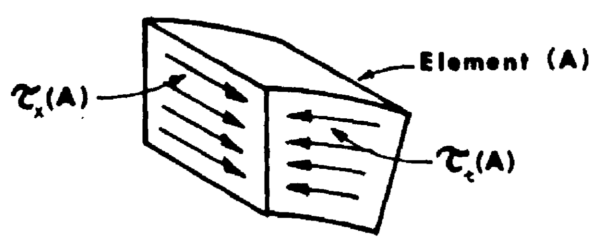
The torsional shear stress distribution on any cross section of a circular bar will vary linearly along any radial line emanating from the geometric centroid, and will have the same distribution on all radial lines (Fig. B8.2.1-1A). The longitudinal shear stress (τ_x), which is equal to the torsional shear stress (τ_t), produces no warping of the cross section when the stress distribution is the same on adjacent radial lines (Figs. B8.2.1-1B and B8.2.1-1C). For noncircular sections, the torsional shear stress distribution is nonlinear (except



A. Circular Bar Shear Stress Distribution



B. Differential Element Location



C. Differential Element

FIGURE B8.2.1-1 SHEAR STRESS DISTRIBUTION along lines of symmetry where the cross section contour is normal to the radial line) and will be different on adjacent radial lines (Fig. B8.2.2-2A). When the torsional and longitudinal shear stress is different on adjacent radial lines, warping of the cross section will occur (Fig. B8.2.2-2B).

When the warping deformation induced by longitudinal shear stresses is restrained, normal stresses (σ) are induced to maintain equilibrium. These normal stresses are neglected in the torsional analysis of solid sections since they are small, attenuate rapidly, and have little effect on the angle of twist.

Restraints to the warping deformation occur at fixed ends and at points where there is an abrupt change in the applied twisting moment.

II Limitations

The torsional analysis of solid cross sections is subject to the following limitations.

A. The material is homogeneous and isotropic

B. The shear stress does not exceed the shearing proportional limit and is proportional to the shear strain (elastic analysis).

C. The stresses calculated at points of constraint and at abrupt changes of applied twisting moment are not exact.

D. The applied twisting moment cannot be an impact load.

E. The bar cannot have an abrupt change in cross section.*

III Membrane Analogy

The torsional analysis of solid bars with irregularly shaped cross sections is usually complex, and for some cases unsolvable. The membrane analogy can be used to visualize the solution for these cross sections.

The basic differential equation for a torsional analysis, written in terms of the St. Venant's stress function, is:

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* Stress concentration factors must be used at abrupt changes in the cross section.

$$\frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = -2G\theta .$$

This equation is similar to the basic differential equation used for the analysis of a deflected membrane, which is:

$$\frac{\partial^2 \mu}{\partial y^2} + \frac{\partial^2 \mu}{\partial z^2} = -p/T .$$

The following analogies exist between the solutions of these two analyses when the membrane has the same boundaries as the cross section of a twisted bar.

A. The volume under the deflected membrane for any pressure (p) is equal to one-half the applied twisting moment (M_t), when $2G\theta = p/T$ numerically.

B. The tangent to a contour line of deflected membrane at any point is in the same direction as the maximum torsional shear stress at the same point on the cross section.

C. The slope at any point in the deflected membrane normal to the contour at that point is proportional to the magnitude of the torsional shear stress at that point on the cross section.

IV Basic Torsion Equations for Solid Sections

A. Torsional Shear Stress

The basic equation for determining the torsional shear stress at an arbitrary point (P) on an arbitrary cross section is:

$$\tau_t = \frac{M(x)}{S_t}$$

where M(x) is evaluated at $x = L_x$ for the arbitrary cross section where the torsional shear stress is to be determined.

If a constant torque is applied to the end of the bar, the equation reduces to:

$$\tau_t = \frac{M_t}{S_t} .$$

S_t will vary along the length of the bar for a varying cross section and, in this case, the equation is:

$$\tau_t = \frac{M(x)}{S_t(x)} .$$

For the case of varying moment and varying cross section, both M(x) and $S_t(x)$ must be evaluated at the cross section where the torsional shear stress is to be determined.

In the equations for torsional shear stress determinations in sections B8.2.2-III through B8.2.2-VIII, M(x) is equal to M_t and the stress is determined at the point of maximum torsional shear stress. The resulting equations determine maximum shear stress only.

B. Angle of Twist

The basic equation for determining the angle of twist between the origin and any cross section located at a distance L from the origin is:

$$\phi = \frac{1}{GK} \int_0^{L_x} M(x)\, dx.$$

When M(x) is a constant torsional moment applied at the end of the bar, the equation reduces to:

$$\phi = \frac{1}{GK} \int_0^{L_x} M_t \, dx = \frac{M_t L_x}{GK} ,$$

and the total twist of the bar is:

$$\phi\,(\max) = \frac{M_t L}{GK} .$$

When the cross section varies along the length of the bar, the torsional constant becomes a function of x and must be included within the integral as follows:

$$\phi = \frac{1}{G} \int_0^{L_x} \frac{M(x)}{K(x)} \, dx .$$

The moment-area technique (numerical integration) is very useful in calculating angle of twist between any two sections when a $M(x)/GK(x)$ -Diagram is used. See Section B8.2.3, example problem 3.

B8.2.2 TORSIONAL SHEAR STRESS AND ANGLE OF TWIST FOR SOLID SECTIONS

The equations presented in this section for torsional shear stress are for points of maximum torsional shear stress. For some cross sections, torsional shear stress equations are presented for more than one location.

The equations presented for angle of twist are for the total angle of twist developed over the full length of the bar.

The applied load in all cases is a concentrated twisting moment (M_t) applied at the end of the bar.

I Circular Section

The maximum torsional shear stress occurs at the outside surface of the circular cross section (Fig. B8.2.1-1A) and is determined by

$$\tau_t(\max) = \frac{M_t}{S_t}$$

where

$$S_t = \frac{J}{\rho} = \frac{\pi r^4}{2}\left(\frac{1}{r}\right) = \frac{\pi r^3}{2} \quad .$$

Since the torsional shear stress varies linearly from the centroid of the section, the stress at any point (P) on the cross section is determined by

$$\tau_t(P) = \frac{M_t \rho}{J} \quad .$$

The total angle of twist is determined by

$$\phi(\max) = \frac{M_t L}{GK}$$

where

$$K = \frac{\pi r^4}{2} .$$

The angle of twist between the origin and an arbitrary cross section located at a distance L_x from the origin is determined by

$$\phi = \frac{M_t L_x}{GK} .$$

II <u>Hollow Circular Section</u>

The torsional shear stress and angle of twist for a thick-walled hollow cylinder can be determined from the equations in Section B8.2.2.-I when the torsional constant and the torsional section modulus are determined by the following equations.

$$K = \frac{\pi}{2}\left(r_o^4 - r_i^4\right)$$

$$S_t = \frac{\pi}{2}\left(r_o^3 - r_i^3\right)$$

where r_o and r_i are defined by Figure B8.2.2-1.

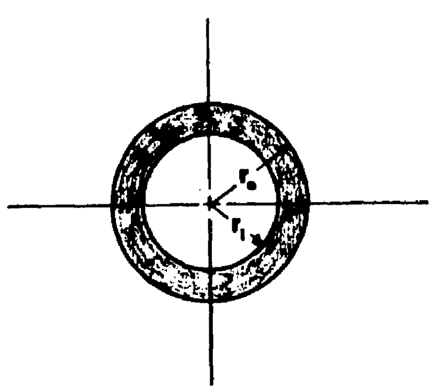


FIGURE B8.2.2-1 HOLLOW CIRCULAR CROSS SECTION

III Rectangular Section

The maximum torsional shear stress occurs at point A (Fig. B8.2.2-2A) and is determined by the following equation.

$$\tau_t\,(\max) = M_t/S_t$$

where

$$S_t = \alpha\, b\, d^2 \; .$$

Some typical values of α are shown in Table B8.2.2-1.

The equation for α in terms of (b/d) is

$$\alpha = \frac{1}{\left[3.0 + \frac{1.8}{(b/d)}\right]} \; .$$

The torsional shear stress at point (B) is determined by

$$\tau_t(B) = \tau_t(\max)\left(\frac{b}{d}\right) .$$

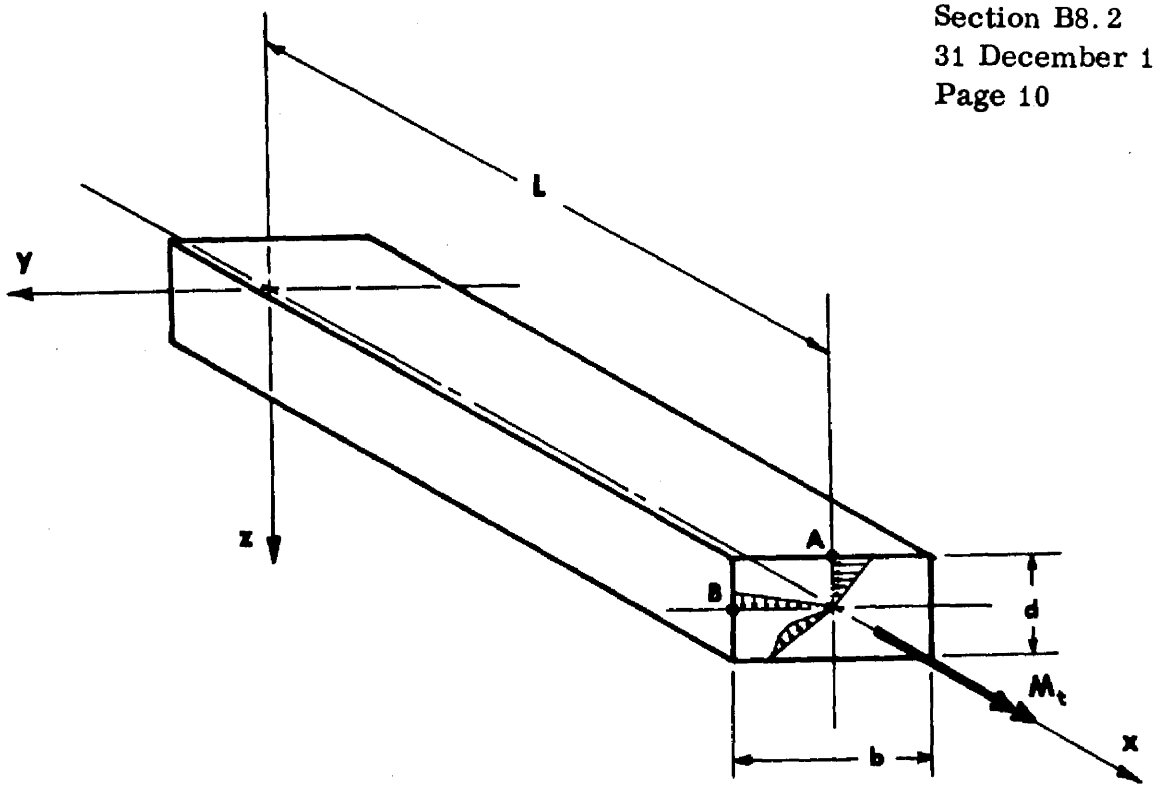
The total angle of twist is determined by

$$\phi\;(\max) = \frac{M_t L}{GK}$$

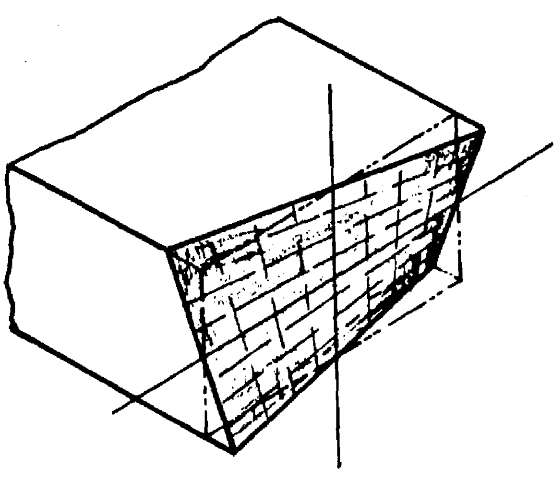
where

$$K = \beta\, b\, d^3 \; .$$

Some typical values of β are shown in Table B8.2.2-1.



A. Stress Distribution on Rectangular Cross Section



B. Warping Deformation of a Rectangular Cross Section

FIGURE B8.2.2-2 RECTANGULAR CROSS SECTION

The equation for β in terms of (b/d) is

$$\beta = \left[0.333 - \frac{0.21}{(b/d)}\left(1.0 - \frac{0.0833}{(b/d)^4}\right)\right] .$$

TABLE B8.2.2-1

b/d	1.0	1.5	2.0	2.5	3.0	4.0	6.0	10.0	100	∞
α	0.208	0.238	0.256	0.269	0.278	0.290	0.303	0.314	0.331	0.333
β	0.141	0.195	0.229	0.249	0.263	0.281	0.298	0.312	0.331	0.333

The stress distributions on different radial lines are shown in Figure B8.2.2-2A, and the resulting warping deformation at an arbitrary cross section located at distance L_x from the origin is shown in Figure B8.2.2-2B.

IV <u>Elliptical Section</u>

The maximum torsional shear stress occurs at point A (Fig. B8.2.2-3) and is determined by

$$\tau_t(\max) = M_t/S_t$$

where

$$S_t = \frac{\pi bd^2}{16} .$$

The torsional shear stress at point B is determined by

$$\tau_t(B) = \tau_t(\max)\left(\frac{b}{d}\right) .$$

The total angle of twist is determined by the following equation:

$$\phi(\max) = \frac{M_t L}{KG}$$

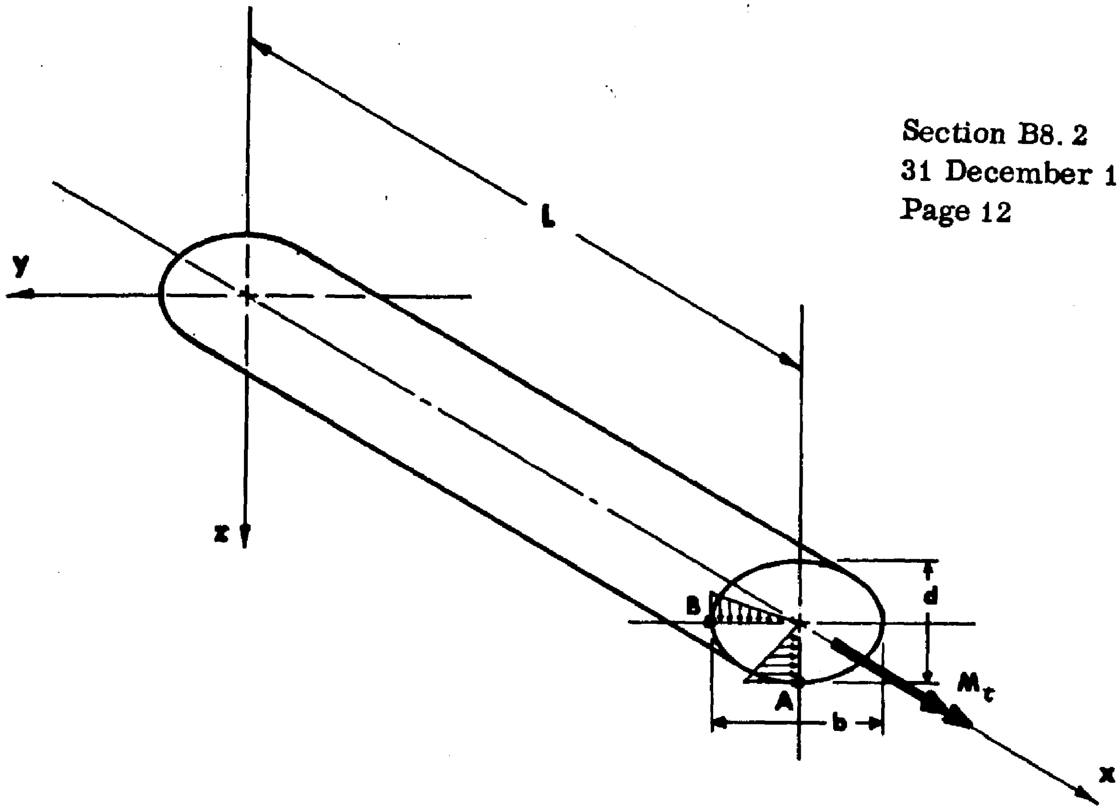


FIGURE B8.2.2-3 ELLIPTICAL CROSS SECTION

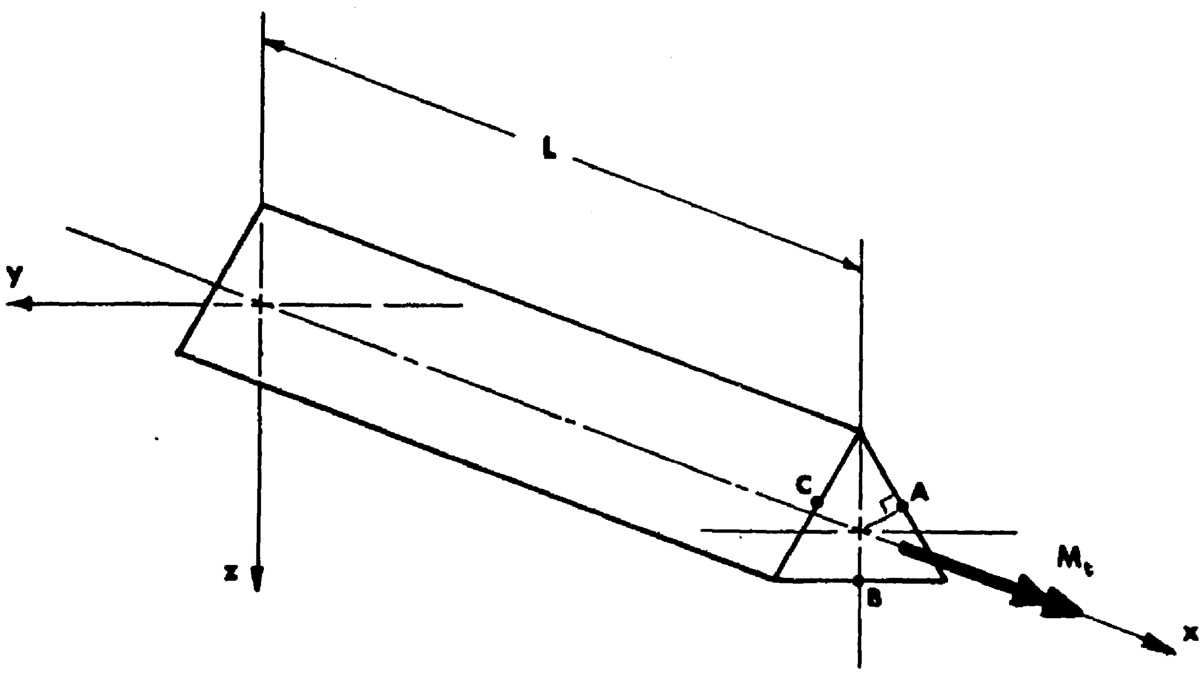


FIGURE B8.2.2-4 EQUILATERAL TRIANGULAR CROSS SECTION where

$$K = \frac{\pi b^3 d^3}{16 (b^2 + d^2)} .$$

V Equilateral Triangular Section

The maximum torsional shear stress occurs at points A, B, and C (Figure B8.2.2-4) and is determined by

$$\tau_t(\max) = \frac{M_t}{S_t}$$

where

$$S_t = b^3/20.$$

The total angle of twist is determined by

$$\phi \max = \frac{M_t L}{KG}$$

where

$$K = \frac{b^4 \sqrt{3}}{80} .$$

VI Regular Hexagonal Section

The approximate maximum torsional shear stress is determined by

$$\tau_t(\max) = M_t/S_t$$

where

$$S_t = 0.217Ad$$

and is located at the midpoints of the sides (Fig. B8.2.2-5). A is the cross-sectional area and d is the diameter of the inscribed circle.

The approximate total angle of twist is determined by

$$\phi\ (\max) = \frac{M_t L}{KG}$$

where

$$K = 0.133Ad^2\ .$$

VII <u>Regular Octagonal Section</u>

The approximate maximum torsional shear stress is determined by

$$\tau_t(\max) = M_t/S_t$$

where

$$S_t = 0.223Ad$$

and is located at the midpoints of the sides (Fig. B8.2.2-6). A is the cross-sectional area and d is the diameter of the inscribed circle.

The approximate total angle of twist is determined by

$$\phi\ (\max) = \frac{M_t L}{KG}$$

where

$$K = 0.130Ad^2\ .$$

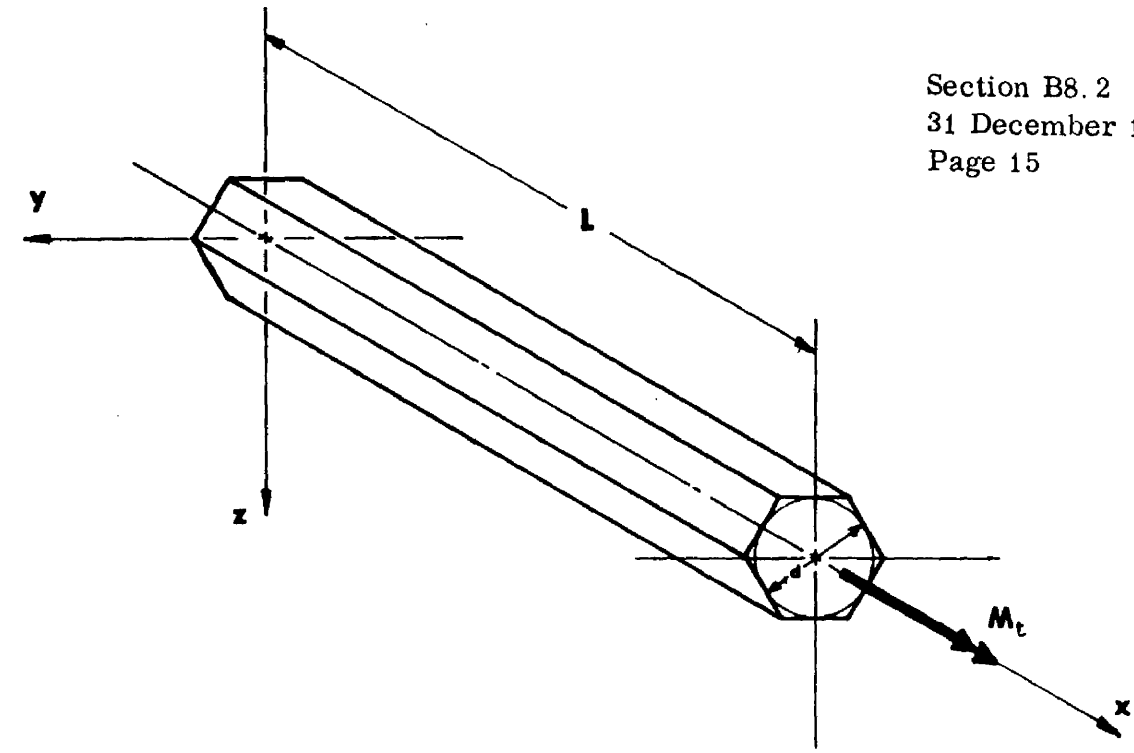


FIGURE B8.2.2-5 REGULAR HEXAGONAL CROSS SECTION

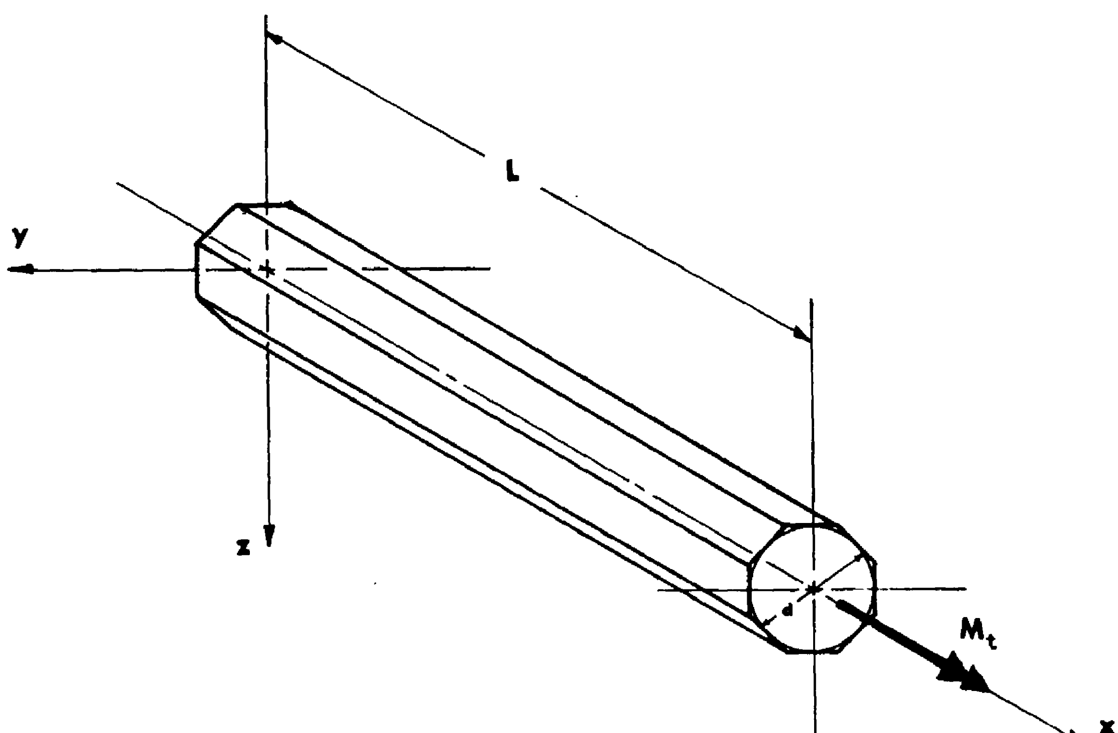


FIGURE B8.2.2-6 REGULAR OCTAGONAL CROSS SECTION

VIII Isosceles Trapezoidal Section

The approximate maximum torsional shear stress and total angle of twist can be determined for an isosceles trapezoid when the trapezoid is replaced by an equivalent rectangle. The equivalent rectangle is obtained by drawing perpendiculars to the sides of the trapezoid (CB and CD) from the centroid C and then forming rectangle EFGH using points B and D (Fig. B8.2.2-7).

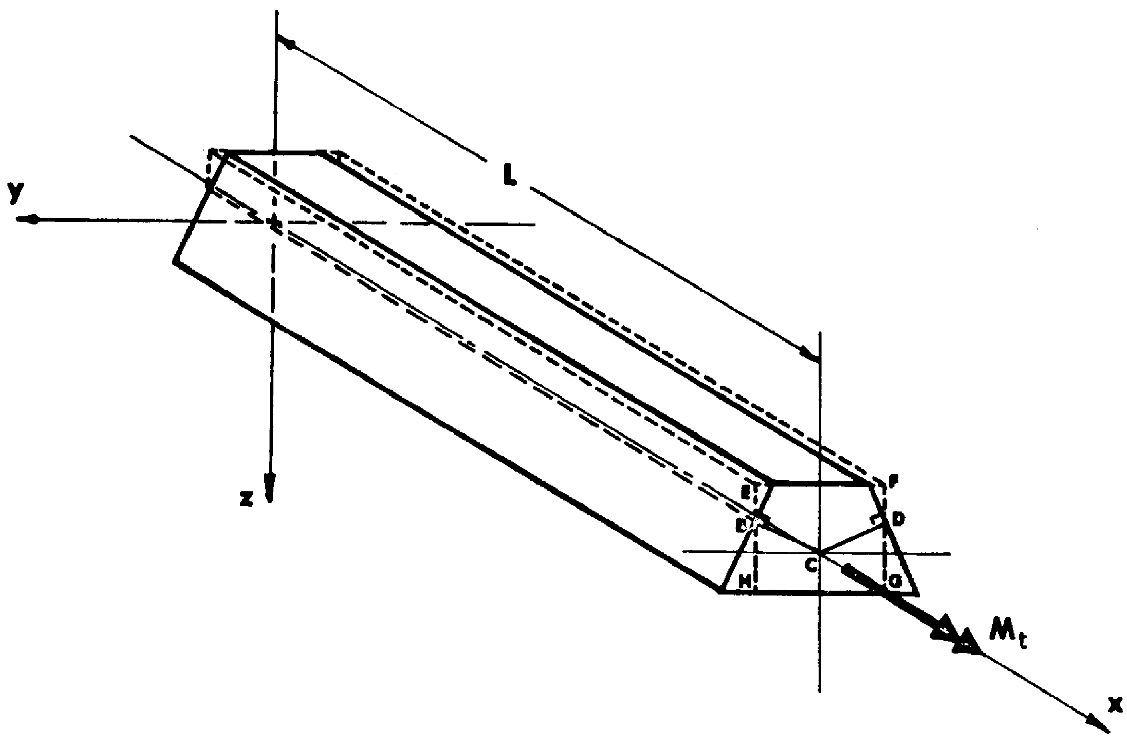


FIGURE B8.2.2-7 ISOSCELES TRAPEZOIDAL SECTION

B8.2.3 EXAMPLE PROBLEMS FOR TORSION OF SOLID SECTIONS

I Example Problem 1

Find the maximum torsional shear stress and the total angle of twist for the solid rectangular cross section shown in Figure B8.2.3-1.

Solution: From Table B8.2.2-1, $\alpha = - 0.256$ and $\beta = 0.229$ for b/d = 2.0. The maximum stress will occur at point A in Figure B8.2.3-1.

The torsion section modulus (S_t) is

$$S_t = \alpha b d^2$$

$$= 0.256\,(5)(2.5)^2$$

$$= 8.00 \text{ in}^3$$

The torsional shear stress (τ_t) at point A is

$$\tau_t = M_t/S_t$$

$$= 100,000/8.00$$

$$\tau_t = 12,500 \text{ psi.}$$

The torsional constant (K) is

$$K = \beta b d^2$$

$$= 0.229\,(5)(2.5)^2$$

$$= 7.156 \text{ in}^4 .$$

The total angle of twist (ϕ) is

$$\phi = M_t L/GK$$

$$= 100,000\,(32)/4,000,000\,(7.156)$$

$$= 0.1118 \text{ rad.}$$

II Example Problem 2

Find the maximum torsional shear stress (τ_t) and the angle of twist (ϕ) at point B for the tapered bar shown in Figure B8.2.3-2 with a constant distributed torque.

Solution:

The radius (r) of the tapered bar as a function of the x-coordinate is:

$$r = 2.5 - 0.005x$$

The internal twisting moment M(x) as a function of the x-coordinate is:

$$M(x) = m_t x = 200x \ .$$

The torsional section modulus (S_t) of the bar as a function of the x-coordinate is

$$S_t(x) = 0.5 \ \pi r^3$$

$$= 1.5708 \ [2.5 - 0.005x]^3$$

The maximum torsional shear stress (τ_t) at point B is

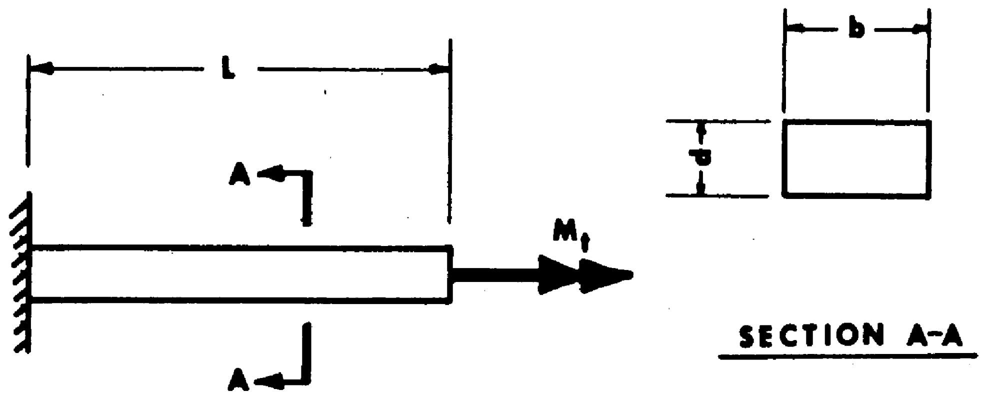
$$\tau_t = \frac{M(x)}{S_t(x)}$$

$$= \frac{200x}{1.5708 \ (2.50 - 0.005x)^3} \quad .$$

For $x = L_x = 300$,

$$\tau_t = 38,197 \text{ psi} \quad .$$

Since both the internal twisting moment and the torsional stiffness vary with the x-coordinate, the angle of twist is obtained from

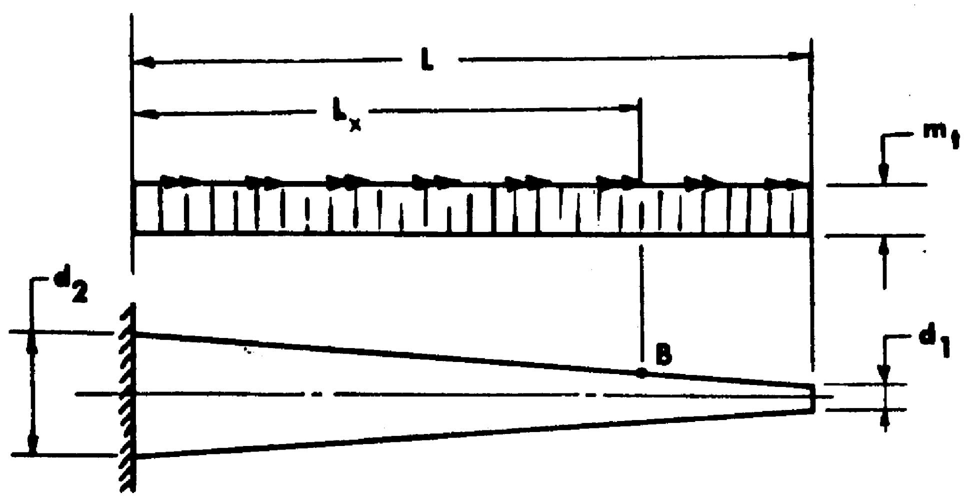


b = 5.0"
d = 2.5"

M_t = 100,000 in. lb.
G = 4,000,000 psi.

L = 32"

FIGURE B8.2.3-1



L = 400"
L_x = 300"

d_1 = 1"
d_2 = 5"

m_t = 200 in. lb./in.
G = 4,000,000 psi.

FIGURE B8.2.3-2

$$\phi = \frac{1}{G} \int_0^{L_x} \frac{M(x)}{K(x)} \, dx \; .$$

The torsional stiffness (K) of the bar as a function of the x-coordinate is

$$K = 0.5 \, \pi r^4$$

$$= 1.5708 (2.5 - 0.005x)^4 \; .$$

The angle of twist (ϕ) in radians between the origin and point B can be obtained by evaluating the following integral.

$$\phi = \frac{1}{4x10^6} \int_0^{300} \frac{200x}{(2.5 - 0.005x)} \, dx \; .$$

III Example Problem 3

Find the total angle of twist (ϕ) at points A, B, C, and D for the bar loaded as shown in Figure B8.2.3-3.

Solution:

The internal moments at points A, B, C, and D are:

$$M(D) = 0$$

$$M(C) = \int_{35}^{50} \left(6,670 - \frac{x}{7500}\right) dx = 15.0 \text{ in-Kips}$$

$$M(B) = 15,000 + \int_{20}^{35} 1000 \, dx = 30.0 \text{ in-Kips}$$

$$M(A) = 30,000 + \int_0^{20} 3000 \, \frac{x}{20} \, dx = 60.0 \text{ in-Kips} \; .$$

The equation for angle of twist is

$$\phi = \int_0^{L_x} \frac{M_t}{GK} dx,$$

which is also the area under the M_t/GK diagram (Fig. B8.2.3-4).

Using the moment-area analogy from beam theory, the following statement can be made.

"The total angle of twist between any two cross sections of a bar which is loaded with an arbitrary torsional load is equal to the area under the M_t/GK diagram between the two cross sections."

Using the moment-area principle above, the angles of twist are

ϕ (A) = 0 fixed end.

$$\phi\,(B) = 30{,}000(20) + 2/3(30{,}000)(20) = \frac{1{,}000{,}000}{GK} = 0.100 \text{ rad}$$

$$\phi\,(C) = 1{,}000{,}000 + 15{,}000(15) + 1/2\,(15{,}000)(15) = \frac{1{,}337{,}500}{GK}$$

$$= 0.13375 \text{ rad}$$

$$\phi\,(D) = 1{,}337{,}500 + 1/3\,(15{,}000)(15) = \frac{1{,}412{,}500}{GK} = 0.14125 \text{ rad.}$$

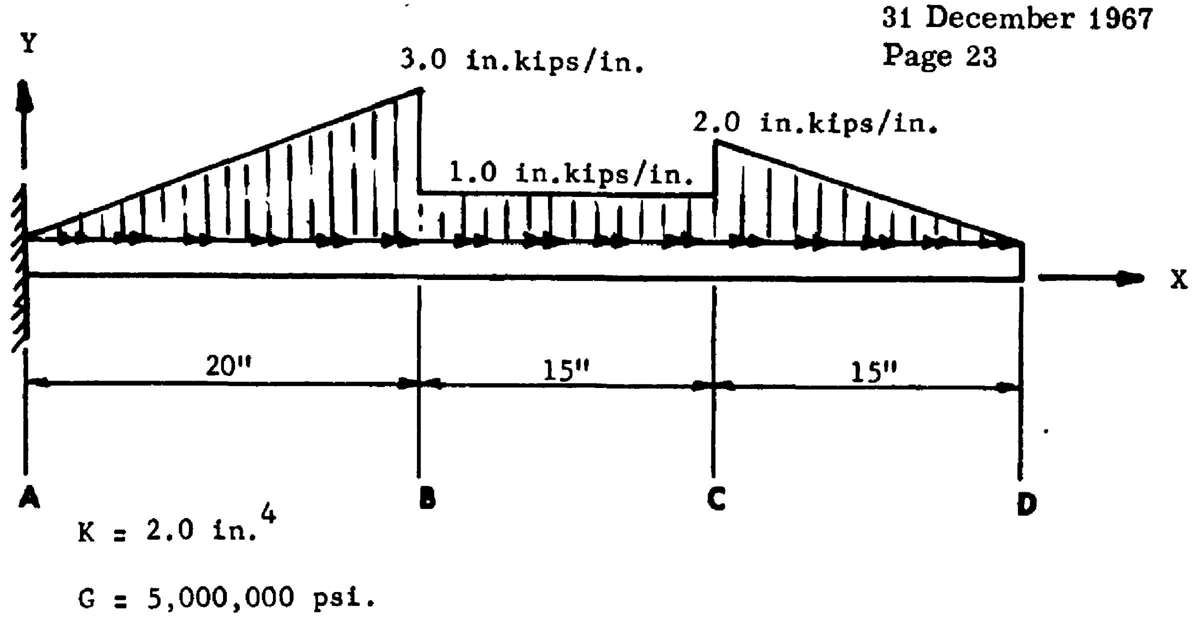


FIGURE B8.2.3-3

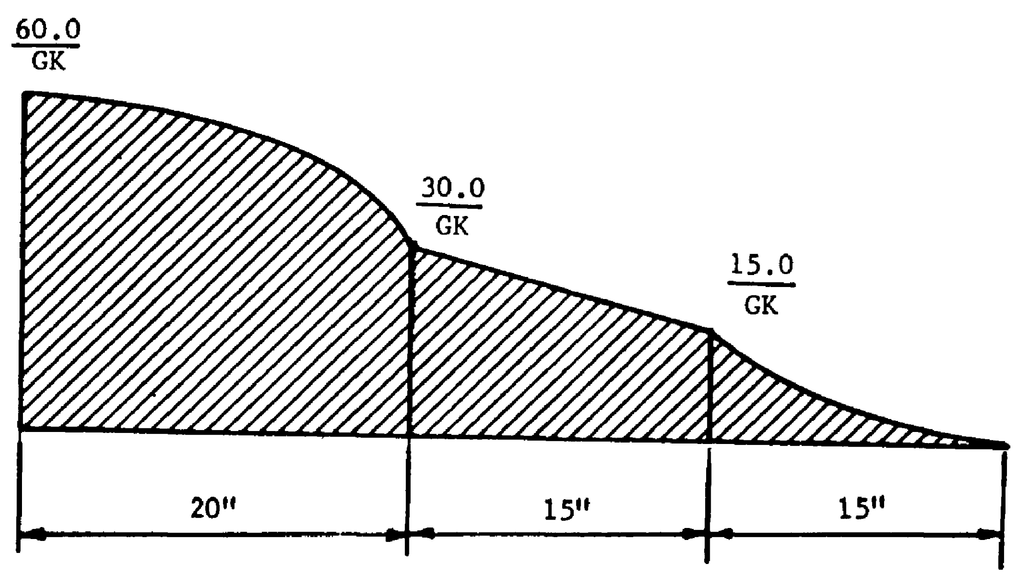


FIGURE B8.2.3-4