B7.3.4 STIFFENED SHELLS

Up to this point, only homogeneous, isotropic, monocoque shells have been considered.

It is known that certain rearrangements of the material in the section increase the rigidity; consequently, less material is needed, and this affects the efficiency of design. Therefore, to obtain a more efficient and economical structure, the material in the section should be arranged to make the section most resistant to certain predominant stresses. Based on this premise, stiffened structures were developed.

B7.3.4.1 General

Stiffened shells are commonly used in the aerospace and civil engineering fields. The shell functions more efficiently if the meridional system, circumferential system, or a combination of both systems of stiffeners is used. The meridional stiffeners usually have all the characteristics of beams and are designed to take compressional and bending influences more effectively than the monocoque section. The circumferential stiffeners provided most of the lateral support for the meridional stiffeners. However, circumferential stiffeners are capable of withstanding moments, shears, and axial stresses.

If the stiffeners are located relatively close together, it appears logical to replace the stiffened section with an equivalent monocoque section having the corresponding ideal modulus of elasticity. Then the shell under discussion can be analyzed as a monocoque shell. More details on this approach will be given in later sections. The geometry included is for cylindrical, spherical, and conical shells.

I Cylindrical Shell

This shell may have longitudinal stiffening, circumferential stiffening, or both. Stiffening may be placed on the internal or external side of the surface, or it may be located on both sides. If cut-outs are needed, they will usually be located between the stiffeners.

II Spherical Shell

This shell, if stiffened, will usually be stiffened in both meridional and circumferential directions. The problem may be slightly more complicated in the meridional direction because, obviously, the section that corresponds to this direction will decrease in size toward the apex. This leads to the non-uniform ideal thickness.

III Conical Shell

This configuration structurally lies between cases I and II.

IV Approach for Analysis

The approach for analysis is similar for all shells. If only circumferential stiffening exists, the structure can be cut into simple elements consisting of cylindrical, conical, or spherical elements and rings as shown in Figure B7.3.4-1 and, considering the primary loading, the interaction will be performed as given in Paragraph B7.3.2. If only longitudinal stiffeners are present, interaction of cylindrical panels with longitudinal beams (stiffeners) will be performed, as shown in Figure B7.3.4-2.

If both circumferential and longitudinal stiffeners are present, the panel will be supported on all four sides. The ratio of circumferential to longitudinal distances between the stiffeners is very important. These panels loaded with pressure (external or internal) will transmit the reactions to the circumferential and longitudinal stiffeners.

There are no fixed formulas in existence for stiffened shells in general. If the stiffeners are close together, the structure can be analyzed as a shell. Then the stiffened section, for the purpose of analysis, should be replaced with the equivalent monocoque section, which is characterized with the equivalent modulus of elasticity. This replacement has to be done for both meridional and circumferential directions. Both sections will possess ideal monocoque properties, the same thickness, but different ideal moduli of elasticity. This leads to the idea of orthotropic material. The concept of orthotropy will be studied in detail in a later section, and a proper analysis procedure will be suggested.

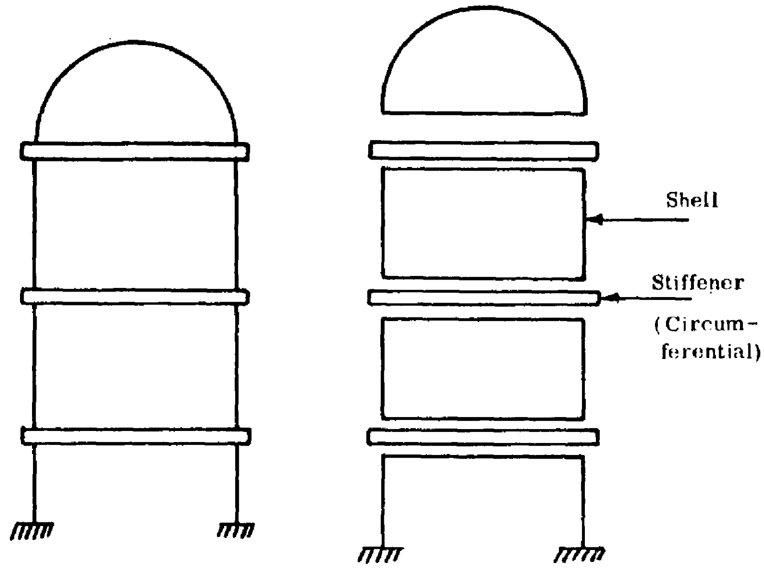


FIGURE B7. 3. 4-1 CIRCUMFERENTIALLY STIFFENED SHELL

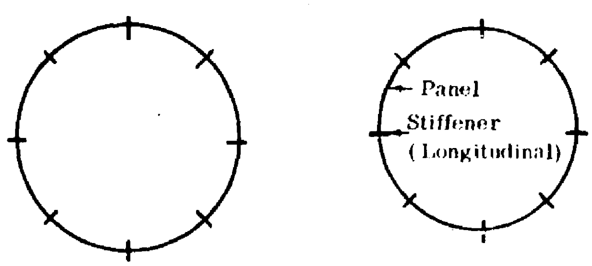


FIGURE B7. 3. 4-2 LONGITUDINALLY STIFFENED SHELL

V Method of Transformed Section

This approximate method covers all variations of stiffened (and sandwich) construction, regardless of the kind of elements that make up the section. This method shows how the combined section can be substituted with an equivalent monocoque section of the same stiffness. This idealized section should be determined for the circumferential and meridional directions of the shell. Then the analyst deals with an orthotropic, monocoque shell. The analysis of orthotropic shells is similar to the analysis of monocoque shells discussed previously, if certain corrections are entered into the formulas cited at that time. The analysis for shells where the shear distortions cannot be neglected is more complicated and will be explained in detail in the following sections.

Assume a composite section (stiffened, sandwich, etc.) which consists of different layers of material, as shown in Figure B7.3.4-3. Each layer (i) is characterized by a modulus of elasticity (E_i) and a cross-sectional area (A_i). First select one convenient modulus of elasticity (E^*) as a basis for the equivalent monocoque section which is to be established. Accordingly, all layers will be modified and reduced to one equivalent material which is characterized with E^*. In this manner, the ideal transformed section (Figure B7.3.4-4) is determined. It should be noted that, for the convenience of design, the thickness (t_i) of individual layers was not changed, but areas A_i become A_i^*. The same modulus of elasticity (E^*) now corresponds to every A_i^*, thus making the entire section homogeneous.

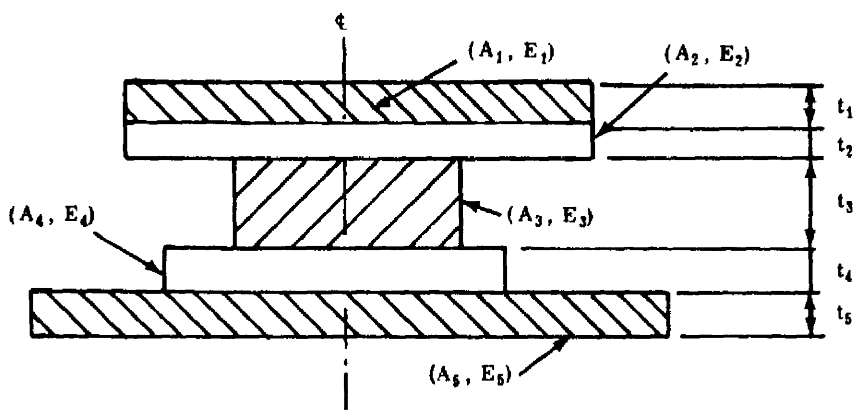


FIGURE B7.3.4-3 ORIGINAL COMPOSITE SECTION

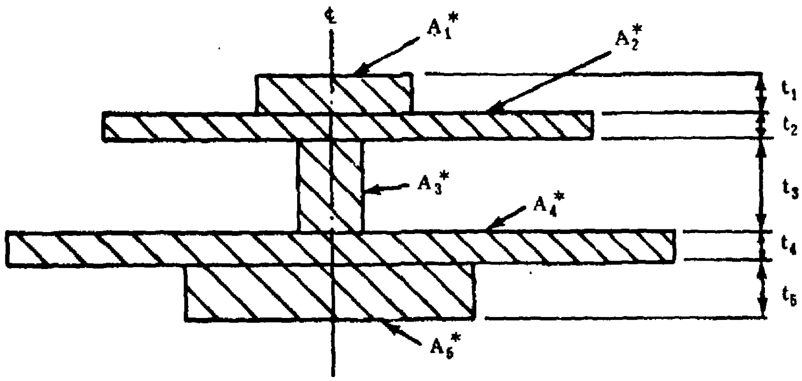


FIGURE B7. 3. 4-4 TRANSFORMED SECTION

The necessary computations are presented in Table B7. 3. 4-1. Designations are given on the sketch included in the table.

The computations lead to the determination of the moment of inertia of an equivalent section. The ideal monocoque rectangular section can be determined as having the same bending resistance as the original section. For example, if the section is symmetrical about the neutral axis, the thickness (t) can be found for the new monocoque rectangular section of the same resistance as follows:

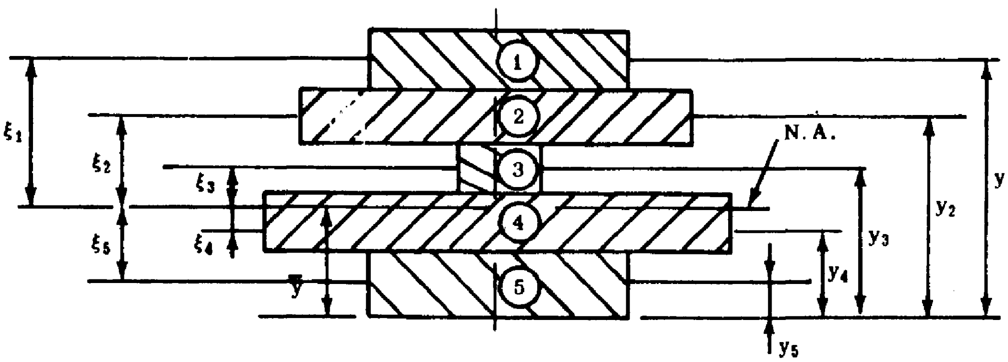
$$I = \frac{bt^3}{12} = I^* ; \qquad t = 2.29 \sqrt[3]{\frac{I^*}{b}}$$

where b is the selected width of the new section.

B7. 3. 4. 2 Sandwich Shells

The basic philosophy which the analyst applies to a sandwich structure is precisely the same as he would apply to any structural element. This procedure consists of determining a set of design allowables with which the set of applied loads is compared.

TABLE B7.3.4-1 TRANSFORMED SECTION METHOD



Element	A_i	E_i	$n_i = \frac{E_i}{E^*}$	$A_i^* = n_i A_i$	y_i	$A_i^* y_i$	ξ_i	$A_i^* \xi_i^2$	$A_i^* t_i^2$
①									
②									
③									
·									
·									
·									
				ΣA_i^*		$\Sigma A_i^* y_i$		$\Sigma A_i^* \xi_i^2$	$\Sigma A_i^* t_i^2$

$$\bar{y} = \frac{\Sigma A_i^* y_i}{\Sigma A_i}$$

$$I^* = \Sigma A_i^* \xi^2 + \frac{1}{12} \Sigma A_i^* t_i^2$$

Generally, two types of "allowables" data exist. The first type is determined by simple material tests and is associated with material more than with geometry, and the second type is dependent upon the geometry of the element. If, in a sandwich construction, the materials of construction are considered to be the core, facings, and bonding media, the basic material properties would be associated with the properties of these three independent elements.

The second class of allowables data is distinguished by being dependent upon configuration as well as upon the basic properties of the facings, core, and bond media. This class of failure modes may be further subdivided into modes of failure that include the entire configuration, and those modes that are localized to a portion of the structure but still limit the overall load-carrying capacity.

The most important local modes of failure are dimpling, wrinkling, and crimping. These modes of failure are dependent upon the local geometry and upon the basic properties of the materials of the sandwich. The general modes of failure generally are associated with the buckling strength of sandwich structural elements. This will be discussed in Section B7. 4.

In this paragraph, the general design of sandwich shells under pure static conditions will be presented. Two fundamental cases will be recognized:

1. Shear deformations can be neglected.

2. Shear deformations are extensive; however, shear can be taken by the core. No new basic theories are required, only the application of established theory. Once the shear deformation is properly included in the analysis, the analysis is complete.

The first logical approximation would be to replace actual sandwich with orthotropic material. The concepts of orthotropy actually may cover not only the large family of sandwiches, but also other materials such as corrugated shells, etc.

To give a systematical description of orthotropic analysis, attention will again be directed to the mathematical structure of the analytical formulas for the monocoque shells presented earlier in this section. This will make clear what kind of modifications can be made to apply the same formulas (that were derived for monocoque material) to the orthotropic shells.

B7.3.4.3 Orthotropic Shells

A material is orthotropic if the characteristics of the materials are not the same in two mutually orthogonal directions (two-dimensional space). Such material has different values for E, G, and μ for each direction. Poisson's ratio, μ, also may be different in the case of bending and axial stresses. In the majority of cases this difference is negligible, but to distinguish one from the other, μ will be designated for Poisson's ratio which corresponds to bending stresses, and μ' for axial stresses. The behavior of the shell under loading is a function of certain constants that depend upon the previously mentioned material constants and geometry. The special case of orthotropy is isotropy (the material characteristics in both directions of two-dimensional space are the same). To see the dependence of stresses and deformations in shells upon previously mentioned constants, a short review of isotropic concepts of shells is provided. These constants are designated with extensional and bending rigidities.

I Extensional and Bending Rigidities

In the past, only isotropic monocoque shells were considered, and numerous formulas were presented. The definitions for isotropic shells are as follows:

Extensional rigidity

$$B = \frac{Et}{1 - \mu^2}$$

Bending rigidity

$$D = \frac{Et^3}{12(1 - \mu^2)}$$

B and D have appeared in many previous formulas.

The following characteristic stress formulas apply for rotationally symmetric thin shells:

$$N_\phi = B\left(\epsilon_\phi + \mu\epsilon_\theta\right)$$

$$N_\theta = B\left(\epsilon_\theta + \mu\epsilon_\phi\right) \quad .$$

The bending loads are

$$M_\phi = -D\left(\frac{1}{R_\phi}\ \frac{d}{d\phi} + \mu\frac{\beta}{R_\theta}\cot\phi\right)$$

$$M_\theta = D\left(\frac{\beta}{R_\theta}\cos\phi + \mu\frac{1}{R_\phi}\ \frac{d\beta}{d\phi}\right) \quad .$$

The final stresses can be obtained as follows:

$$\sigma_\phi = \frac{E}{1-\mu^2}\left(\frac{N_\phi}{B} + \frac{M_\phi}{D}\,z\right)$$

$$\sigma_\theta = \frac{E}{1-\mu^2}\left(\frac{N_\theta}{B} - \frac{M_\theta}{D}\,z\right) \quad . \qquad (1)$$

For a monocoque section of rectangular shape

$$\sigma_\phi = \frac{N_\phi}{t} + \frac{M_\phi}{\frac{t^3}{12}}\,z$$

$$\sigma_\theta = \frac{N_\theta}{t} - \frac{M_\theta}{\frac{t^3}{12}} \quad . \qquad (2)$$

The physical meaning of D and B is obvious if equations (1) and (2) are compared.

The componental stresses due to membrane forces are

$$1\sigma_{\phi} = \frac{EN_{\phi}}{(1 - \mu^2)B} = \frac{E}{1 - \mu^2} \cdot N_{\phi} \frac{1 - \mu^2}{Et} = \frac{N_{\phi}}{1 \times t} = \frac{N_{\phi}}{A}$$

$$1\sigma_{\theta} = \frac{EN_{\theta}}{(1 - \mu^2)B} = \frac{E}{1 - \mu^2} \cdot N_{\theta} \frac{1 - \mu^2}{Et} = \frac{N_{\theta}}{1 \times t} = \frac{N_{\theta}}{A}$$

where

$$A = 1 \times t \quad .$$

It is convenient to choose the width of the section strip that is equal to unity.

The componental stresses due to bending:

$$2\sigma_{\phi} = \frac{M_{\phi} z}{D} \cdot \frac{E}{1 - \mu^2} = M_{\phi} z \frac{E}{(1 - \mu^2)} \frac{12(1 - \mu^2)}{Et^3} = \frac{12M_{\phi} z}{1 \times t^3} = \frac{M\phi z}{I}$$

$$2\sigma_{\theta} = \frac{M_{\theta} z}{D} \cdot \frac{E}{1 - \mu^2} = M_{\theta} z \frac{E}{(1 - \mu^2)} \frac{12(1 - \mu^2)}{Et^3} = \frac{12M_{\theta} z}{1 \times t^3} = \frac{M_{\phi} z}{I}$$

where

$$I = \frac{1 \times t^3}{12} \quad .$$

Evidently, if stiffened or sandwich shell is being used, a modified B and D shall be used in the equations, then all previously derived equations for monocoque shells may be used for stiffened and sandwich shells. If the values from the "transformed section" are used, then

$$B = \frac{A^* E^*}{1 - \mu^2} \quad ; \qquad D = \frac{I^* E^*}{1 - \mu^2} \quad .$$

In the preceding formulas $\mu' \approx \mu$ is assumed.

II Orthotropic Characteristics

Now the orthotropy is defined if, for two mutually orthogonal main directions, 1 and 2, the following constants are known or determined:

D_1, B_1, μ_1, μ_1' and shear rigidity D_{Q_1}

D_2, B_2, μ_2, μ_2' and shear rigidity D_{Q_2} .

To use the previously given formulas (for the isotropic case) for the analysis of the orthotropic structures, the formulas must be modified. For this purpose, a systematical modification of the primary and secondary solutions will be provided in the following paragraph to make possible the use of the unit-edge loading method for the orthotropic case. In the analysis of monocoque shells, the shear distortions usually are neglected. With sandwich, in most cases, such neglect is justified. The previously collected formulas for the isotropic case do not include the shear distortion. Consequently, orthotropic analyses which neglect the shear distortions will be examined first. Later, an additional study will be presented which considers the distortions due to the shear.

III Orthotropic Analysis, If Shear Distortions Are Neglected

A. Primary Solutions

It was previously stated that in most cases, the primary solutions are membrane solutions. For the purpose of interaction, the following set of values is needed (considering axisymmetrically loaded shells of revolution).

N_θ - membrane load in circumferential direction

N_ϕ - membrane load in meridional direction u - displacement in the direction of tangent to the meridian

w - displacement in the direction of the normal-to-the-middle surface.

Actually, having u and w, any componental displacements can be obtained from the pure geometric relations if only the axisymmetrical cases are considered. Consequently, for this purpose, it will be adequate to investigate u and w.

To determine N_θ and N_ϕ, all formulas that were presented for the isotropic case can be used, because the membrane is a statically determinate system and independent of the material properties.

When N_θ and N_ϕ are obtained, u and w can be obtained in the following manner.

First determine the strains components (ϵ_ϕ and ϵ_θ). For the isotropic case, the correspondent formulas are

$$\epsilon_\phi = \frac{1}{Et} (N_\phi - \mu' N_\theta)$$

$$\epsilon_\theta = \frac{1}{Et} (N_\theta - \mu' N_\phi) \quad .$$

For the orthotropic case the same formulas may be written

$$\epsilon_\phi = \frac{1}{B_\phi (1 - \mu_\phi' \mu_\theta')} (N_\phi - \mu_\theta' N_\theta)$$

$$\epsilon_\theta = \frac{1}{B_\theta (1 - \mu_\theta' \mu_\phi')} (N_\theta - \mu_\phi' N_\phi) \quad .$$

Note: $D = B \frac{t^2}{12}$.

Displacement can now be obtained from the following differential equation:

$$\frac{du}{d\phi} - u \cot \phi = R_1 \epsilon_\phi - R_2 \epsilon_\theta = f(\phi) \quad .$$

The solution of the equation above is

$$u = \sin \phi \left(\int \frac{f(\phi)}{\sin \phi} \, d\phi + C \right)$$

where C is the constant of integration to be determined from the condition at the support. Then, the displacement (w) is obtained from the following equation:

$$\epsilon_\theta = \frac{u}{R_2} \cot \phi - \frac{w}{R_2} \quad .$$

Consequently, for every symmetrically loaded shell of revolution the stresses and deformations are determined for the orthotropic case.

B. Secondary Solutions

To obtain the secondary solutions, the formulas that were derived for the isotropic case can be used and then, using the substitution of proper constants, they can be transformed into formulas for the orthotropic case. Generally, due to any edge disturbance (unit loading) the formulas give direct solutions for

$$N_\phi, \; N_\theta, \; M_\phi, \; M_\theta, \; Q, \; \beta, \text{ and } \Delta r$$

in the form of:

Solution = (edge disturbance) × (function of significant constant) × (function of geometry).

Cylindrical Shell - All formulas given in Tables B7.3.3-9 and B7.3.3-10 can be modified if the following replacement is made:

$$k = \frac{L}{\sqrt{Rt}} \sqrt[4]{3(1 - \mu^2)} \rightarrow \frac{L}{2R} \sqrt{\frac{B_y\left(1 - \mu_x \mu_\theta\right)}{D_x}}$$

$$D = \frac{Et^3}{12(1 - \mu^2)} \rightarrow D_x = \frac{E_x t^3}{12(1 - \mu_1 \mu_2)}$$

$$B = \frac{Et}{1 - \mu^2} \rightarrow B_y = \frac{E_y t}{1 - \mu_x' \mu_y'}$$

$$D = B \frac{t^2}{12} .$$

E_x = the modulus of elasticity in longitudinal direction.

E_y = the modulus of elasticity in circumferential direction.

IV Orthotropic Analysis, If Shear Distortions Are Included

For this more complicated case, the solution may be found in Reference 4, which was considered as the basis for Paragraphs IV and V. Cylinders and spheres only are are considered herein.

A. Cylindrical Shell

In the case of a cylinder constructed from a sandwich with relatively low traverse shear rigidity, the shear distortion may not be negligible; therefore, an analysis is presented which includes shear distortion for a symmetrically loaded orthotropic sandwich cylinder.

The following nomenclature is used:

D_x, D_y = Flexural stiffnesses of the shell wall per inch of width of orthotropic shell in axial and circumferential directions, respectively (in.-lb).

D_{Q_x} = Shear stiffness in x-z plane per inch of width (lb/in.)

B_x, B_y = Extensional stiffnesses of orthotropic shell wall in axial and circumferential directions, respectively (lb/in.).

M_x = Moment acting in x direction (in.-lb/in.).

Q_x = Transverse shear force acting in x-z plane (lb/in.).

μ_x, μ_y = Poisson's ratio associated with bending in x and y directions, respectively.

μ_x', μ_y' = Poisson's ratio associated with extension in x and y directions, respectively.

The derived solutions are presented in Tables B7.3.4-2 and B7.3.4-3.

B. Half Spheres

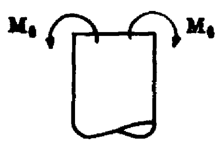
Based on Geckler's assumption for the half sphere, all formulas derived for cylinders can be adapted for the spherical shell as well.

V Influence of Axial Forces on Bending in Cylinder

Usually it is assumed that the contribution of the axial force (N_0) to the bending deflection is negligible; however, for a cylinder with a relatively large radius, the axial force may significantly contribute to the bending deflection. Therefore, the preceding analysis was extended by the same author (Reference 4) to include the effect of the axial force on the deflections. This leads to modification of the formulas (Tables B7.3.4-2 and B7.3.4-3) in the manner shown in Table B7.3.4-4. The constants are slightly modified as follows:

$$\alpha^2 = \frac{\left[\frac{B_y\left(1 - \mu_x'\mu_y'\right)}{D_{Q_x} R^2} + \frac{N_0}{D_x}\right]}{4\left[1 + \frac{N_0}{D_{Q_x}}\right]}$$

TABLE B7.3.4-2 INFLUENCE COEFFICIENTS DUE TO EDGE LOADING M_0 IN ORTHOTROPIC CYLINDER



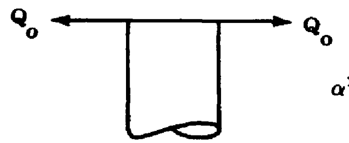
Constants

$$\alpha^2 = \frac{B_y\left(1-\mu'_x\mu'_y\right)}{4D_{Q_x}R^2} \qquad \beta^4 = \frac{B_y\left(1-\mu'_x\mu'_y\right)}{4D_xR^2} \qquad D = D_x$$

	$\alpha^4 - \beta^4 > 0$	N^{0*}	$\alpha^4 - \beta^4 = 0$	N^{0*}	$\alpha^4 - \beta^4 < 0$	N^{0*}
w	$\frac{M_0}{D(A_2-A_1)}\left[\frac{A_2}{4\alpha^2-A_1^2}e^{-A_1x}+\frac{A_1}{A_2^2-4\alpha^2}e^{-A_2x}\right]$	1	$\left[\frac{e^{mx}}{2\alpha^2}+\frac{xe^{mx}}{m}\right]\left(-\frac{M_0}{D}\right)$	4	$\frac{e^{-Vx}}{2D\beta^4}M_0\left(-\beta^2\cos Px+\frac{V}{P}\beta^2\sin Px\right)$	7
M_x	$M_0\left(\frac{A_2e^{-A_1x}-A_1e^{-A_2x}}{A_2-A_1}\right)$	2	$e^{mx}M_0(1-mx)$	5	$\frac{e^{-Vx}}{P}M_0(P\cos Px+V\sin Px)$	8
Q_x	$M_0\left(\frac{-A_1A_2e^{-A_1x}+A_1A_2e^{-A_2x}}{A_2-A_1}\right)$	3	$-2\alpha^2e^{mx}xM_0$	6	$-\frac{e^{-Vx}}{P}M_0\sin Px(V^2+\mu^2)$	9
Constants:	$A_1=\left[2\alpha^2+2(\alpha^4-\beta^4)^{1/2}\right]^{1/2}$ $A_2=\left[2\alpha^2-2(\alpha^4-\beta^4)^{1/2}\right]^{1/2}$		$m=-(2)^{1/2}\alpha$		$V=(\beta^2+\alpha^2)^{1/2}$ $P=(\beta^2-\alpha^2)^{1/2}$	

	Influence Coefficients for the Displacements and Rotation at the Edge of the Shell	
$w_x = 0$	$-M_0/2\beta^2D$	10
$\left[\frac{dw}{dx}\right]_{x=0}$	$(M_0/D^2)(\alpha^2+\beta^2)^{1/2}$	11

TABLE B7.3.4-3 INFLUENCE COEFFICIENTS DUE TO EDGE LOADING Q_0 IN ORTHOTROPIC CYLINDER



Constants

$$\alpha^2 = \frac{B_y\left(1 - \mu'_x \mu'_r\right)}{4D_{Q_x} R^2}\ ; \qquad \beta^4 = \frac{B_y\left(1 - \mu'_x \mu'_y\right)}{4D_x R^2}\ ; \qquad D = D_x$$

	$\alpha^4 - \beta^4 > 0$	N_0*	$\alpha^4 - \beta^4 = 0$	N_0*	$\alpha^4 - \beta^4 < 0$	N_0*
w	$\frac{Q_0}{D(A_2 - A_1)}\left[\frac{e^{-A_1x}}{4\alpha^2 - A_1^2} + \frac{e^{-A_2x}}{A_2^2 - 4\alpha^2}\right]$	12	$\frac{Q_0}{D\alpha^2}\left[\frac{e^{mx}}{m} + \frac{xe^{mx}}{2}\right]$	15	$\frac{e^{-Vx}}{2D\beta^4} Q_0 \left(\frac{\alpha^2}{P} \sin Px - V \cos Px\right)$	18
M_x	$Q_0\left[\frac{e^{-A_1x} - e^{-A_2x}}{A_2 - A_1}\right]$	13	$e^{mx}\, x Q_0$	16	$\frac{e^{-Vx}}{P} Q_0 \sin Px$	19
Q_x	$Q_0\left[\frac{-A_1e^{-A_1x} + A_2e^{-A_2x}}{A_2 + A_1}\right]$	14	$e^{mx} Q_0\left[1 + mx\right]$	17	$\frac{e^{-Vx}}{P} Q_0 \left[P \cos Px - V \sin Px\right]$	20
Constants	$A_1 = \left[2\alpha^2 + 2(\alpha^4 - \beta^4)^{1/2}\right]^{1/2}$ $A_2 = \left[2\alpha^2 - 2(\alpha^4 - \beta^4)^{1/2}\right]^{1/2}$		$m = -(2)^{1/2}\alpha$		$V = (\beta^2 + \alpha^2)^{1/2}$ $P = (\beta^2 - \alpha^2)^{1/2}$	
Influence Coefficients for the Displacements and Rotation at the Edge of the Shell						
$w_{x=0}$	$-Q_0 \frac{(\beta^2 + \alpha^2)^{1/2}}{2\beta^4 D}$					21
$\left[\frac{dw}{dx}\right]_{x=0}$	$Q_0 \frac{2\alpha^2 + \beta^2}{2D\beta^4}$					22

TABLE B7. 3. 4-4 MODIFICATION OF TABLES B7. 3. 4-2 AND -3 TO INCLUDE THE EFFECTS OF AXIAL FORCES ON BENDING

Formula	Quantities: (Formulas in Tables B7.3.4-2 and -3)	Substitute
1	$(4\alpha^2 - A_1^2)$ $(A_2^2 - 4\alpha^2)$	$(4\gamma^2 - A_1^2)$ $(A_2^2 - 4\gamma^2)$
4	Whole Formula	$w = \frac{e^{mx} M_0}{D(m^2 - 4\gamma^2)^2} \left[(4\gamma^2 - 3m^2) + (m^2 - 4\gamma^2) mx\right]$
6	$2\alpha^2 M_0 x$	$m^2 M_0$
7	Whole Formula	$w = \frac{M_0 e^{-Sx}}{2VD} \left[(S^2 + \gamma^2) \cos Px + \frac{S}{P} (\gamma^2 + P^2) \sin Px\right]$
8	V	S
9	V	S
10	Whole Formula	$w_{x=0} = (-M_0/2VD)\ (2\alpha^2 + \beta^2 - 2\gamma^2)$
11	Whole Formula	$\left[\frac{dw}{dx}\right]_{x=0} = (M_0\beta^2/VD)\ (\alpha^2 + \beta^2)^{1/2}$
12	$(4\alpha^2 - A_1^2)$ $(A_2^2 - 4\alpha^2)$	$(4\gamma^2 - A_1^2)$ $(4_2^2 - 4\gamma^2)$
15	Whole Formula	$w = \frac{e^{mx} Q_0}{D(m^2 - 4\gamma^2)^2} \left[2m - (m^2 - 4\gamma^2) x\right]$
18	Whole Formula	$w = \frac{Q_0 e^{-Sx}}{2VD} \left[\frac{\gamma^2}{P} \sin Px - S \cos Px\right]$
19	V	S
20	V	S
21	Whole Formula	$(-Q_0/2VD)\ (\alpha^2 + \beta^2)^{1/2}$
22	Whole Formula	$\left(\frac{dw}{dx}\right)_{x=0} = \frac{Q_0}{2VD}\ (\beta^2 + 2\gamma^2)$

$$\beta^4 = \frac{B_y\left(1 - \mu'_x \mu'_y\right)}{4 D_x R^2 \left[1 + \frac{N_0}{D_{Q_x}}\right]}$$

$$\gamma^2 = \frac{B_y\left(1 - \mu'_x \mu'_y\right)}{4 D_{Q_x} R^2} \quad ; \quad V = 4\gamma^4 - 4\gamma^2 \alpha^2 + \beta^4$$

$$S = (\beta^2 + \alpha^2)^{1/2}$$

$$D = D_x$$

B7. 3. 5 UNSYMMETRICALLY LOADED SHELLS

Until now, the axisymmetrical cases have been treated with respect to the geometry, material, and loading. The "unit-load method" was exclusively used for the solution. It was shown that the most complex solutions are applied to the shells without symmetry, loaded unsymmetrically. The first level of simplification of the complex procedures would be the usage of axisymmetric shell loaded unsymmetrically.

The scope of this manual does not permit presentation of the actual derivations, but solutions for the most commonly appearing cases in engineering will be presented in the remaining tables.

The shells are assumed to be thin enough to use the membrane theory. These tables of solutions also provide the necessary information about the loading and geometry.

B7. 3. 5. 1 Shells of Revolution

The first level of simplification of the complex procedures would be axisymmetric shells loaded unsymmetrically. Similarly, symmetrical shells may have unsymmetrical boundaries, which will cause the symmetrical loading to be no longer symmetrical.

Table B7. 3. 5-1 presents some solutions for certain loadings for spherical, conical, and cylindrical shells loaded unsymmetrically.

B7. 3. 5. 2 Barrel Vaults

This paragraph presents the collection of different solutions for curved panels of simple beam system. The geometry of curved panels is circular, elliptical, cycloidal, parabolic, catenary, and special shape. The solutions for different loadings are given in Tables B7. 3. 5-2, B7. 3. 5-3, and B7. 3. 5-4. The shells under consideration are thin, and linear theory was the basis for the derived formulas.

TABLE B7.3.5-1 SPHERE, CONE, AND CYLINDER LOADED BY WIND LOADING

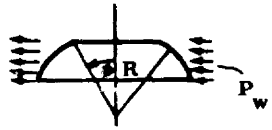
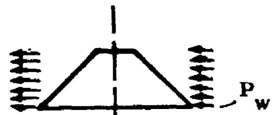
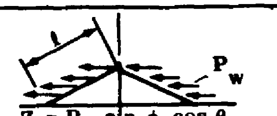
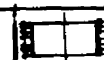
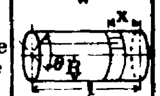
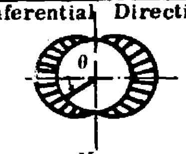
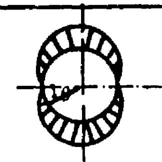
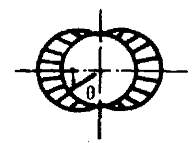
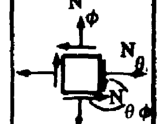
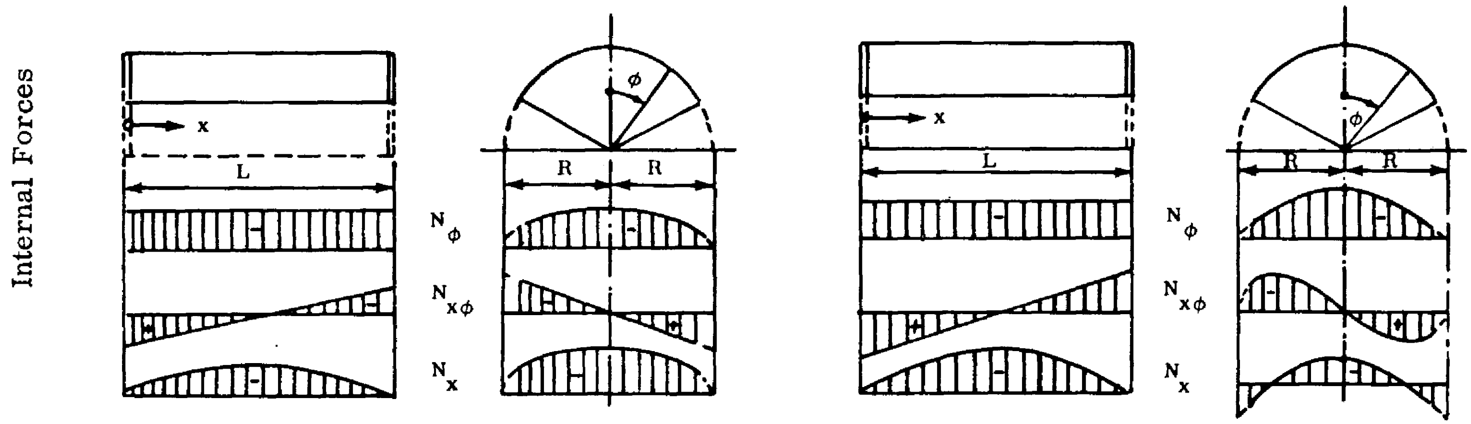
	Sphere	Cone	Conical Shell	Cylinder
	 $Z = P_w \sin\phi \cos\theta$	 $Z = P_w \sin\phi \cos\theta$	 $Z = P_w \sin\phi \cos\theta$ Conical Shell Supported at the Vertex of the Cone with Free Edge	 $Z = P_w \cos\theta$ 
	Circumferential Direction  N_ϕ	 $N_{\theta\phi}$	 N_θ	 Differential Element
N_ϕ	$-p_w \frac{R}{3} \frac{\cos\theta \cos\phi}{\sin^3\phi} \left[3(\cos\phi_0 - \cos\phi) - (\cos^3\phi_0 - \cos^3\phi)\right]$ For $\phi_0 = 0$ (No Opening) $-p_w \frac{R}{3} \frac{\cos\theta \cos\phi}{\sin^3\phi} (2 - 3\cos\phi + \cos^3\phi)$	$-p_w \frac{x}{2} \left[\cos\phi - \frac{1}{3\cos\phi} - \frac{x_0^2}{x^2} \left(\cos\phi - \frac{1}{\cos\phi}\right) - \frac{x_0^3}{x^3} \frac{2}{3\cos\phi}\right] \cos\theta$ For $\phi_0 = 0$ (Complete Cone) $-p_w \frac{x}{2} \left(\cos\phi - \frac{1}{3\cos\phi}\right) \cos\theta$	$p_w \left[\frac{\ell^3 - x^3}{3x^2} - \frac{\ell^2 - x^2}{2x} \sin^2\phi\right] \frac{\cos\theta}{\cos\phi}$	$p_w \frac{x^2}{2R} \cos\theta$
$N_{\phi\theta}$	$-p_w \frac{R}{3} \frac{\sin\theta}{\sin^3\phi} \left[3(\cos\phi_0 - \cos\phi) - \cos^3\phi_0 + \cos^3\phi\right]$ For $\phi_0 = 0$ (No Opening) $-p_w \frac{R}{3} \frac{\sin\theta}{\sin^2\phi} (2 - 3\cos\phi + \cos^3\phi)$	$-p_w \frac{x^3 - x_0^3}{3x^2} \sin\theta$ For $x_0 = 0$ (Complete Cone) $-p_w \frac{x}{3} \sin\theta$	$p_w \frac{\ell^3 - x^3}{3x^2} \sin\theta$	$-p_w x \sin\theta$
N_θ	$p_w \frac{R}{3} \frac{\cos\theta}{\sin^3\phi} \left[\cos\phi(3\cos\phi_0 - \cos^3\phi_0) - 3\sin^2\phi - 2\cos^4\phi\right]$ For $\phi_0 = 0$ (No Opening) $p_w \frac{R}{3} \frac{\cos\theta}{\sin^3\phi} (2\cos\phi - 3\sin^2\phi - 2\cos^4\phi)$	$-p_w x \cos\phi \cos\theta$ For $x_0 = 0$ (Complete Cone) $-p_w x \cos\phi \cos\theta$	$-p_w x \cos\phi \cos\theta$	$-p_w R \cos\theta$

TABLE B7.3.5-2 CURVED CIRCULAR PANELS, BARREL VAULTS

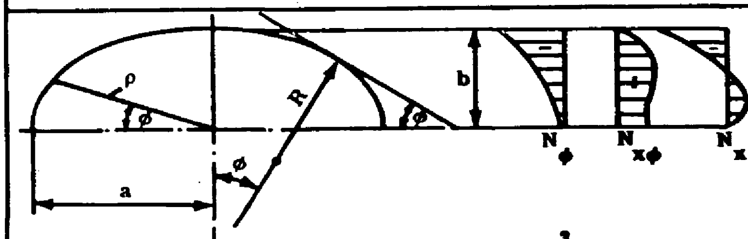
Deadweight Loading (q)		Uniform Load on Projected Area
	$X = 0$	
	$Y = q \sin \phi$	Loading $X = 0$, $Y = p \sin \phi \cos \phi$
	$Z = q \cos \phi$	$Z = p \cos^2 \phi$



N_ϕ	$-qR \cos \phi$	$-pR \cos^2 \phi$	At $\phi = 45°$
$N_{x\phi}$	$q(L - 2x) \sin \phi = N_{\phi x}$	$N_{\phi x} = \frac{3}{4} p(L - 2x) \sin 2\phi$	$N_x = 0$
N_x	$-\frac{qx}{R} (L - x) \cos \phi$	$-\frac{3p}{2R} \times (L - x) \cos 2\phi$	$N_{x\phi} = \max$

TABLE B7. 3. 5-3 CURVED ELLIPTICAL AND CYCLOIDAL PANELS, BARREL VAULTS

Elliptical Panel, Deadweight Loading



The Radius of Curvature: $R = \dfrac{\rho^3}{ab}$

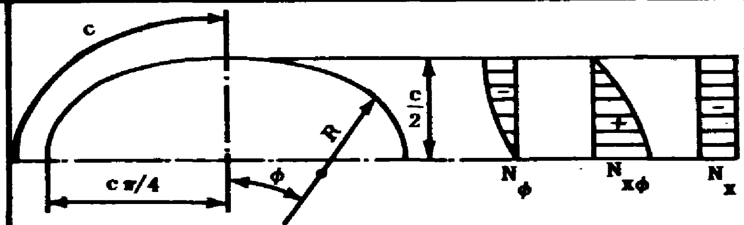
Where $\rho = ab/(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{1/2}$

Loading Components:

$X = O,\ Y = q \sin \phi,\ Z = q \cos \phi$

(For x and L, See Ref. 2)

Cycloidal Panel, Deadweight Loading

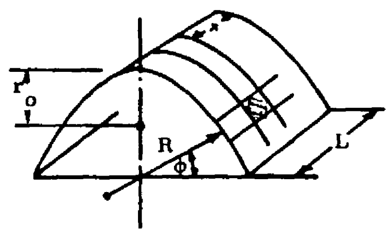


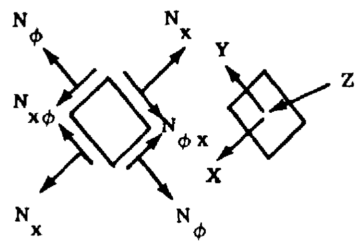
The Radius of Curvature $R = c \cos \phi$

Loading Components: $X = O,\ Y = q \sin \phi,$
$Z = q \cos \phi$

	Elliptical Panel, Deadweight Loading	Cycloidal Panel, Deadweight Loading
N_ϕ	$-\dfrac{q\rho^3}{ab} \cos \phi$	$-q c \cos^2 \phi$
$N_{x\phi}$	$\dfrac{q}{2}(L - 2x)\left(2 + 3\dfrac{a^2 - b^2}{a^2b^2}\rho^2 \cos^2 \phi\right)\sin \phi$	$\dfrac{3}{2} q(L - 2x) \sin \phi$
N_x	$-\dfrac{q}{2}x(L - x)\dfrac{ab}{\rho^3}\left[2 + 3\dfrac{a^2 - b^2}{a^2b^2}\rho^2\left(\cos^2 \phi - 2\dfrac{\rho^2}{b^2}\sin^2 \phi\right)\right]\cos \phi$	$-\dfrac{3q}{2c} x(L - x)$

TABLE B7.3.5-4 BARREL VAULTS





N_x, N_ϕ Unit Normal Forces

$N_{\phi x}$ Unit Central Shear

X, Y, Z Surface Load Components

Boundary Conditions: $N_x = 0$ for $x = 0$ and $x = L$

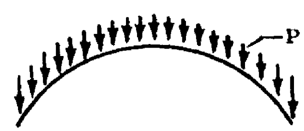
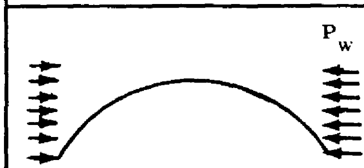
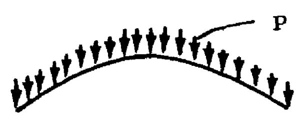
Shell	Loading	M_x	N_ϕ	$N_{\phi x}$
 Circle	$Y = -p\cos\phi$ $Z = p\sin\phi$	$-p\,\frac{x}{R}\,(L-x)\,\sin\phi$	$-pR\sin\phi$	$-p(L-2x)\cos\phi$
 Circle	$Z = p_w\cos\phi$	$-p_w\,\frac{x}{2R}\,(L-x)\,\cos\phi)$	$-p_w R\cos\phi$	$p_w\left(\frac{L}{2}-x\right)\sin\phi$
 Parabola	$Y = -p\cos\phi$ $Z = p\sin\phi$	$p\,\frac{x}{2r_0}\,(L-x)\,\sin^4\phi$	$-p\,\frac{r_0}{\sin^2\phi}$	$p\left(\frac{L}{2}-x\right)\cos\phi$

TABLE B7.3.5-4 BARREL VAULTS (Continued)

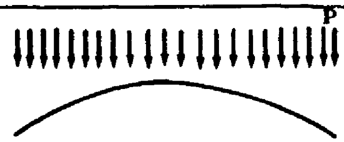
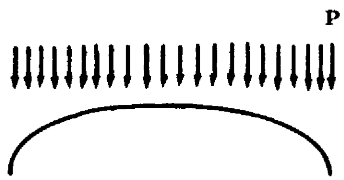
	Shell	Loading	N_x	N_ϕ	$N_{x\phi}$
Parabola		$Y = p \sin\phi \cos\phi$ $Z = p \sin^2\phi$	0	$-p \frac{r_0}{\sin\phi}$	0
Parabola		$Z = p_w \cos\phi$	$p_w \frac{x}{2r_0} (L - x) \sin\phi \cos\phi \times$ $(3 + 2 \sin^2\phi)$	$-p_w r_0 \frac{\cos\phi}{\sin^3\phi}$	$p_w\left(\frac{L}{2} - x\right) \frac{1 + 2\cos^2\phi}{\sin\phi}$
Cycloid		$Y = -p \sin\phi \cos\phi$ $Z = p \sin^2\phi$	$p \frac{2x}{r_0} (L - x) \frac{1 - 2\sin^2\phi}{\sin\phi}$	$-p r_0 \sin^3\phi$	$-p\left(\frac{L}{2} - x\right) 4 \sin\phi \cos\phi$
Cycloid		$Z = p_w \cos\phi$	$-p_w \frac{x}{r_0} (L - x) \times$ $(1 - \cos\phi) \frac{\cos\phi}{\sin^3\phi}$	$-p_w r_0 \sin\phi \cos\phi$	$-p_w\left(\frac{L}{2} - x\right) \frac{1 - 2\sin^2\phi}{\sin\phi}$

TABLE B7. 3. 5-4 BARREL VAULTS (Continued)

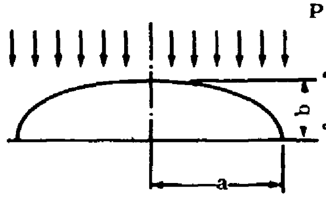
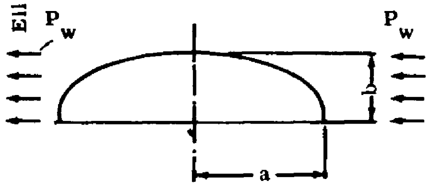
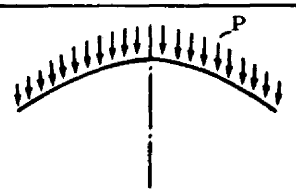
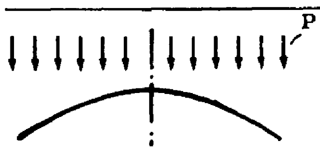
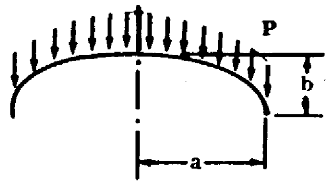
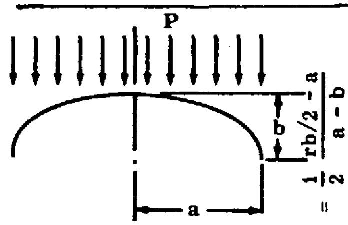
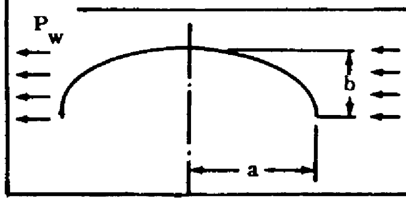
Shell		Loading	N_x	N_ϕ	$N_{x\phi}$
Ellipse	 $\eta = a^2 \cos^2 \phi + b^2 \sin^2 \phi$	$Y = -p \sin \phi \cos \phi$ $Z = p \sin^2 \phi$	$-p \dfrac{3x(L-x)}{2a^2 b^2 \eta^{1/2}} \times \left[b^2(a^2 \cos^2 \phi - b^2 \sin^2 \phi) + 2\eta^2(\sin^2 \phi - \cos^2 \phi)\right]$	$-p a^2 b^2 \dfrac{\sin^2 \phi}{\eta^{3/2}}$	$p\left(\dfrac{L}{2} - x\right) \times \dfrac{3 \sin \phi \cos \phi}{\eta}(b^2 - 2\eta)$
		$Z = p_w \cos \phi$	$-p_w \dfrac{x}{2} \dfrac{(L-x) \cos \phi}{a^2 b^2 \eta^{1/2}} \left[\eta^2 + 3\eta(b^2 - a^2)(1 - 3 \sin^2 \phi) - 6(b^2 - a)^2 \sin^2 \phi \cos^2 \phi\right]$	$-p_w a^2 b^2 \dfrac{\cos \phi}{\eta^{3/2}}$	$p_w\left(\dfrac{L}{2} - x\right) 3 \dfrac{b^2 - a^2}{\eta} \times (1 + \cos^2 \phi) \sin \phi$
Catenary		$Y = -p \cos \phi$ $Z = p \sin \phi$	0	$-p \dfrac{r_0}{\sin \phi}$	0
		$Y = -p \sin \phi \cos \phi$ $X = p \sin^2 \phi$	$p \dfrac{x}{2r_0}(L-x) \sin^2 \phi \times (1 - 2 \sin^2 \phi)$	$-pr_0$	$-p\left(\dfrac{L}{2} - x\right) \sin \phi \cos \phi$

TABLE B7.3.5-4 BARREL VAULTS (Concluded)

Shell	Loading	N_x	N_ϕ	$N_{x\phi}$
Catenary 	$Z = p_w \cos\phi$	$p_w \frac{x}{2r_0}(L - x)$ $(2 + \sin^2\phi)\cos\phi$	$-p_w r_0 \frac{\cos\phi}{\sin^2\phi}$	$p_w\left(\frac{L}{2} - x\right)\frac{1 + \cos^2\phi}{\sin\phi}$
	$Y = -p\cos\phi$ $Z = p\sin\phi$	$p\frac{x(L - x)(\gamma + \pi/4)}{2a(\gamma + \sin\phi)^3}$ $\cdot\left[\gamma(1 - 6\sin^2\phi) -\right.$ $\left.-(2\gamma^2 + 3\sin^2\phi)\sin\phi\right]$	$-p\frac{a}{\gamma + \pi/4}$ $(\gamma + \sin\phi)\sin\phi$	$p\left(x - \frac{L}{2}\right)\frac{(2\gamma + 3\sin\phi)\cos\phi}{\gamma + \sin\phi}$
$r = \frac{a}{\gamma + \pi/4}(\gamma + \sin\phi)$ 	$Y = -p\sin\phi\cos\phi$ $Z = p\sin^2\phi$	$p\frac{x(L - x)(\gamma + \pi/4)}{2a(\gamma + \sin\phi)^3}$ $\cdot\left[\gamma(8 - 15\sin^2\phi)\sin\phi\right.$ $+ (3\gamma^2 + 4\sin^2\phi)$ $\left.\cdot(1 - 2\sin^2\phi)\right]$	$p\frac{a}{\gamma + \pi/4}$ $(\gamma + \sin\phi)\sin^2\phi$	$p\left(x - \frac{L}{2}\right)$ $\frac{(3\gamma + 4\sin\phi)\sin\phi\cos\phi}{\gamma + \sin\phi}$
	$Z = p_w\cos\phi$	$-p_w\frac{x(L - x)(\gamma + \pi/4)}{2a(\gamma + \sin\phi)^3}$ $\cos\phi(1 + \gamma^2 + \sin^2\phi$ $+ 4\gamma\sin\phi)$	$-p_w\frac{a}{\gamma + \pi/4}$ $(\gamma + \sin\phi)\cos\phi$	$p_w\left(x - \frac{L}{2}\right)\frac{1 - 2\sin^2\phi - \gamma\sin\phi}{\gamma + \sin\phi}$

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