SECTION B7.1

MEMBRANE ANALYSIS OF
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B7.1.0.0 MEMBRANE ANALYSIS OF THIN SHELLS OF REVOLUTION

In engineering applications, shells that have the form of surfaces of revolution find extensive application in various kinds of containers, tanks, and domes. Furthermore, this type of shell offers a convenient selection of coordinates.

Thin shells, in general, display large stresses and deflections when subjected to relatively small bending moments. Therefore, in the design of thin shells, the condition of bending stresses is avoided or minimized. If, in the equilibrium equations of such shells, all moment expressions are neglected, the resulting shell theory is called "membrane theory," and the state of stress is referred to as a "momentless" state of stress. There are two types of shells that comply with this membrane theory: (1) shells sufficiently flexible so that they are physically incapable of resisting bending, and (2) shells that are flexurally stiff but loaded and supported in a manner that avoids the introduction of bending strains.

The momentless state of stress in practical shell problems is difficult to achieve. However, with the comparison of the complete bending analysis and membrane analysis for a thin shell of revolution built in along its edges and having noncritical axisymmetric loading, the following conclusions can be made:

1. The stresses and deformations are almost identical for all locations on the shell except for a narrow strip of the shell surface adjacent to the boundary. This narrow strip is usually no wider than $\sqrt{Rt}$.

2. Except for the strip along the boundary, all bending moments, twisting moments, and vertical shears are negligible; this causes the entire bending solution to be practically identical to the membrane solution.
3. Boundary conditions along the supporting edge are very significant; however, local bending and shear decrease rapidly away from the boundary and may become negligible outside the narrow strip. For cases where bending stresses cannot be neglected or when a more complete analysis is desired, see Section B7.3 for bending analysis.

Shells of revolution are frequently loaded internally or externally by forces having the same symmetry as the shell itself. This loading condition is referred to as axisymmetric loading and contributes significantly to the simplification of the analysis methods presented in this section.
B7.1.1.0 GENERAL

Before investigating the stresses and deflections of a shell of revolution, we must examine the geometry of such a surface. A surface of revolution is generated by the rotation of a plane curve about an axis in its plane. This generating curve is called a meridian. The intersections of the generated surface with planes perpendicular to the axis of rotation are parallel circles and are called parallels. For such surfaces, the lines of curvature are its meridians and parallels.

A convenient selection of surface coordinates is the curvilinear coordinate system $\phi$ and $\theta$, where $\phi$ is the angle between the normal to the surface and the axis of rotation and $\theta$ is the angle determining the position of a point on the corresponding parallel, with reference to some datum meridian. (See Figure B7.1.1 - 1.) If the surface of revolution is a sphere, these coordinates are spherical coordinates used in geography; $\theta$ is the longitude and $\phi$ is the complement to the latitude; hence, we have the nomenclature of meridians and parallels.

Figure B7.1.1 - 1 shows a meridian of a surface of revolution. Let $R$ be the distance of one of its points normal to the axis of rotation and $R_1$ its radius of curvature. In future equations, we will also need the length $R_2$, measured on a normal to the meridian between its intersection with the axis of rotation and the shell surface. Noting that $R = R_2 \sin \phi$, the surface of the shell of revolution is completely described by $R_1$ and $R_2$ which are functions of only one of the curvilinear coordinates, $\phi$. $R_0$ will be the radius of curvature when $\phi = 0$. 
Fig. B7.1.1 - 1. Geometry of Surfaces of Revolution
The surface of revolution thus described will be that surface which bisects the thickness of the shell and will henceforth be referred to as the "middle surface" or "reference surface." By specifying the form of the middle surface and the thickness "t" of the shell at any point, the shell is entirely defined geometrically. Figure B7.1.1 - 2 shows an element of the middle surface of the shell.

Fig. B7.1.1 - 2. Shell Element, Middle Surface
When the thickness of the shell is considered for analyzing the internal stresses, it becomes apparent that the radius of curvature "R" cannot be a principal radius of curvature; e.g., it is not normal to the shell surface (except when $R = R_2$ in the special case of circular cylinders). Henceforth, $R_2$ will be used as the principal radius of curvature of an element in the parallel direction. (See Figure B7.1.1 - 3.) The error introduced by this assumption will be negligible in all calculations. Note that $R_1$ is a principal radius of curvature in the meridional direction.

**Fig. B7.1.1 - 3.** Principal Radii of Curvature
Any element of a shell may have the usual internal stresses acting on the faces of the element. These stresses are indicated in Figure B7.1.1 - 4. For the analytical work that will follow, it is convenient to convert these stresses into the resulting forces and moments acting on the middle surface. In the section $\theta = \text{constant}$ (Figure B7.1.1 - 4), the total force normal to this section is by definition $N_{\theta} \, d\phi$. It is the resultant of $\sigma_{\theta}$ stresses acting on this area.

![Shell Stresses Diagram](image)

Because of the curvature of the shell, its width is not simply $d\phi$, but $d\phi \left( \frac{R_1 - z}{R_1} \right)$, and the force transmitted through it is $\sigma_{\theta} \, d\phi \left( 1 - \frac{z}{R_1} \right) \, dz$. 

**Fig. B7.1.1 - 4. Shell Stresses**
The total normal force for the element $ds_\phi$ t is found by integrating from $-\frac{t}{2}$ to $\frac{t}{2}$.

$$N_\theta ds_\phi = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_\theta ds_\phi \left(1 - \frac{z}{R_1}\right) dz$$

When $ds_\phi$ is dropped from both sides, we have the resulting normal force related to the normal stress. In a like manner, $\tau_\theta\phi$ and $\tau_\theta z$ must be integrated to obtain $N_\theta\phi$ and $Q_\theta$. Altogether, we have

$$N_\theta = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_\theta \left(1 - \frac{z}{R_1}\right) dz$$

$$N_\theta\phi = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_\theta\phi \left(1 - \frac{z}{R_1}\right) dz$$

$$Q_\theta = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_\theta z \left(1 - \frac{z}{R_1}\right) dz$$

Applying the same reasoning to the section $\phi$ - constant, we have

$$N_\phi = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_\phi \left(1 - \frac{z}{R_2}\right) dz$$
\[ N_{\phi \theta} = \int_{-t/2}^{t/2} \tau_{\phi \theta} \left( 1 - \frac{z}{R_2} \right) dz \]

and

\[ Q_{\phi} = \int_{-t/2}^{t/2} \tau_{\phi z} \left( 1 - \frac{z}{R_2} \right) dz \, . \]

Note the different radii of curvature for sections $\theta$ constant and $\phi$ constant. (Refer to Figure B7.1.1-2.)

If the stresses are not distributed uniformly across the thickness, bending and twisting moments may result. From $\theta$ constant (Figure B7.1.1-3), the bending moment is

\[ M_{\theta \theta} = \int_{-t/2}^{t/2} \int_{-1/2}^{1/2} \sigma_{\theta \theta} \left( 1 - \frac{z}{R_2} \right) z dz \]

and the twisting moment is

\[ M_{\theta \phi} = \int_{-t/2}^{t/2} \int_{-1/2}^{1/2} \tau_{\theta \phi} \left( 1 - \frac{z}{R_1} \right) z dz \, . \]
In like manner, when $\phi = \text{constant}$,

$$M_{\phi} = \int_{-\frac{t}{2}}^{+\frac{t}{2}} \phi \left(1 - \frac{z}{R_2}\right) zdz$$

$$M_{\phi \theta} = \int_{-\frac{t}{2}}^{+\frac{t}{2}} \tau_{\phi \theta} \left(1 - \frac{z}{R_2}\right) zdz.$$

$N_\theta$, $N_\phi$, $N_{\theta \phi}$, $Q_\phi$, $Q_\theta$, $M_\theta$, $M_\phi$, $M_{\theta \phi}$, and $M_{\phi \theta}$ describe the forces and moments acting on the sides of a rectangular shell element. The fact that the shell element is not necessarily rectangular will be considered when writing the equations of equilibrium in Section B7.1.1.4. Since these ten quantities are all results of stresses, a common name for the group as a whole is "stress resultants." Figure B7.1.1 - 5 shows these stress resultants acting on the middle surface of the shell element. According to membrane theory being considered in the chapter, resultant moments and resultant transverse shearing forces cannot exist. Also, in the assumption of thin shell theory, the quantities $\frac{z}{R_1}$ and $\frac{z}{R_2}$ are very small compared to unity; thus, the only unknowns are the three quantities $N_\theta$, $N_\phi$, and $N_{\theta \phi} = N_{\phi \theta}$. Three equilibrium equations can be written for these three unknowns; hence, the problem becomes statically determinate if the forces acting on the shell are known.
Fig. B7.1.1 - 5. Stress Resultants
B7.1.1.1 NOTATIONS

\( \phi \)  
Angle in vertical plane (measured from axis of rotation) defining the location of a point on the meridian

\( \theta \)  
Angle in horizontal plane that controls the location of a point on the shell

\( R \)  
Radius of a point on the shell measured perpendicular to axis of rotation

\( R_1 \)  
Radius of curvature of meridian at any point

\( R_2 \)  
Radial distance between point on the shell and the axis of rotation

\( R_0 \)  
Radius of curvature when \( \phi = 0 \)

\( t \)  
Shell thickness

\( z \)  
Coordinate in direction of surface normal

\( \sigma_\phi, \sigma_\theta \)  
Internal normal stresses

\( \tau_\theta z, \tau_\theta \phi, \tau_\phi \)  
Inplane shear stresses

\( N_\theta \)  
Circumferential inplane force per unit length at \( \theta \) constant.

\( N_\theta \phi \)  
Shear per unit length acting at \( \theta \) constant

\( Q_\theta \)  
Transverse shear at \( \theta \) constant

\( N_\phi \)  
Meridional inplane force per unit length at \( \phi \) constant

\( N_\phi \theta \)  
Shear per unit length acting at \( \phi \) constant

\( Q_\phi \)  
Transverse shear at \( \phi \) constant
Bending moment per unit length at section
\[ \theta = \text{constant} \]

Twisting moment per unit length at section
\[ \theta = \text{constant} \]

Bending moment per unit length at section
\[ \phi = \text{constant} \]

Twisting moment per unit length at section
\[ \phi = \text{constant} \]

Loading components in radial, circumferential, and meridional directions, respectively

Vertical load

Constant of integration

Angle defining opening in shell of revolution

Displacement in the direction of the tangent to the meridian

Displacement in the vertical direction

Displacement in the direction tangent to parallel

Displacement in the direction normal to surface

Displacement in the horizontal direction

Young's modulus

Poisson's ratio

Strain component in circumferential direction

Strain component in meridional direction

Radius of sphere or major axis length of ellipsoid
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<th>Definition</th>
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<tr>
<td>( \rho )</td>
<td>Specific weight of liquid</td>
</tr>
<tr>
<td>( h )</td>
<td>Height of liquid head</td>
</tr>
<tr>
<td>( b )</td>
<td>Minor axis length of ellipsoid</td>
</tr>
<tr>
<td>( n )</td>
<td>Constant defining the shape of a Cassini dome</td>
</tr>
<tr>
<td>( x )</td>
<td>Coordinate along length of cylinder or along generatrix of cone surface</td>
</tr>
<tr>
<td>( x_0 )</td>
<td>Distance from apex of cone to upper edge of cone measured along generatrix</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>Cone angle</td>
</tr>
<tr>
<td>( s )</td>
<td>Arc length</td>
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B7.1.1.2 SIGN CONVENTIONS

In general, the sign conventions for stresses, displacements, loads, coordinates, etc., are given in the various figures in Section B7.1.1.0. The following is a list of appropriate figures.

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B7.1.1.3 LIMITATIONS OF ANALYSIS

The limitations and assumptions of Section B7.1 are as follows:

1. The analysis is limited to thin shells. A thin shell is usually defined as a shell where the $t/R$ relation can be neglected in comparison to unity. However, this definition is artificial and arbitrary unless those values which are negligible in comparison to unity are defined. For example, if it is assumed that the usual error of five percent is permissible, then the range of thin monocoque shells will generally be dictated by the relation $t/R < 1/20$. The great majority of shells commonly used are in the $1/1000 < t/R < 1/50$ range. This means that they belong to the thin-shell family. If an error of 20 to 30 percent is permissible, the theory of thin shells can be used with caution even when $t/R \leq 1/3$.

2. Flexural strains are zero or negligible compared to direct axial strain.

3. The deflections, rotations, and strains are small. (See Section B7.0 for detailed definition.)

4. The shell is homogeneous, isotropic, and monocoque and is a shell of revolution.

5. It is assumed that Hooke's Law holds (stress is a linear function of strain) and the stresses are within the elastic range.

6. The boundaries of the shell must be free to rotate and to deflect normal to the shell middle surface.
7. Abrupt discontinuities must not be present in shell shape, thickness, elastic constants, or load distribution.

8. Linear elements normal to the unstrained middle surface remain straight during deformation, and their extensions are negligible.

9. Transverse shear strains are zero throughout the thickness.

10. Surface stresses and body forces are negligible.

11. Only nonshallow shells are considered (See Section B7.0.)
B7.1.1.4 EQUATIONS

I GENERAL

The equations presented in this section are for the membrane or
primary solution of the shell. The effects of boundary conditions (secondary
solutions) not compatible with membrane theory will be treated in Section B7.3
on bending theory. Because the bending and membrane theories give practically
the same results except for a strip adjacent to the boundary, the effects of
moments and shears near boundaries can be calculated by using bending theory
and can be superimposed over the membrane solution. The results thus obtained
will be almost identical to those obtained by using the complete, exact bending
theory.

![Boundary Conditions Not Compatible with Membrane Theory](image1)

![Boundary Conditions Compatible with Membrane Theory](image2)

Fig. B7. 1. 1. 4 - 1. Boundary Conditions
B7.1.1.4 EQUATIONS

II EQUILIBRIUM EQUATIONS

The membrane solution is begun by considering the equilibrium of the middle surface of the shell element, cut by two meridians and two parallels (Figure B7.1.1.4 - 2a). The conditions of its equilibrium will furnish three equations in three unknowns, adequate to determine the three unknown stress resultants: the meridional force $N_\phi$, the hoop force $N_\theta$, and the shear $N_{\theta\phi} = N_{\phi\theta}$.

Beginning with the forces parallel to a tangent to the meridian, the shear transmitted by one edge of the element is $N_{\theta\phi} R_1 d\phi$, and on the opposite edge it is \( \left( N_{\theta\phi} + \frac{\partial N_{\theta\phi}}{\partial \theta} d\theta \right) R_1 d\phi \). Only their difference, \( \frac{\partial N_{\theta\phi}}{\partial \theta} R_1 d\theta d\phi \), enters the equilibrium condition. In the same way, we have the difference in the two meridional forces. Bearing in mind that both the force $N_\phi$ and the length $R d\theta$ vary with $\phi$, we have \( \frac{\partial}{\partial \phi} (RN_\phi) d\phi d\theta \). The hoop forces also contribute.

The two forces $N_{\theta} R_1 d\phi$ on either side of the element lie in the plane of a parallel circle where they include an angle $d\theta$. They, therefore, have resultant force $N_{\theta} R_1 d\phi d\theta$ situated in that plane and pointing towards the axis of the shell.

Resolving this force into normal and tangential components shows that $N_{\theta} R_1 d\phi d\theta \cos \phi$ (Figure B7.1.1.4 - 2b) enters the condition of equilibrium.

Finally, considering the component of some external force, $P_\phi RR_1 d\theta d\phi$, the equilibrium equation reads:

\[
\frac{\partial N_{\theta\phi}}{\partial \theta} R_1 d\theta d\phi + \frac{\partial}{\partial \phi} (RN_\phi) d\phi d\theta - N_\theta R_1 d\phi d\theta \cos \phi + P_\phi RR_1 d\theta d\phi = 0 .
\]
Noting that all terms contain \(d\theta d\phi\) gives:

\[
\frac{\partial}{\partial \phi} (RN_\phi) + \frac{\partial N_\phi}{\partial \theta} - R_1 N_\theta \cos \phi + P \phi R_{R_1} = 0.
\]  

(1)

**Fig. B7.1.1.4 – 2. Equilibrium of Shell Element**

By similar reasoning, we obtain an equation for the forces in the direction of the tangent to a parallel circle.
\[
\frac{\partial}{\partial \phi} (RN_{\phi}) + R_1 \frac{\partial}{\partial \theta} R_1 N_{\phi} \cos \phi + P_{\phi} RR_1 = 0. \tag{2}
\]

The third equation is derived from forces perpendicular to the middle surface of the shell.

\[
N_\theta R_1 \sin \phi + N_\phi R - P_z RR_1 = 0.
\]

Dividing by RR_1 and using the geometric relation R = R_2 \sin \phi, we arrive at the third equation of equilibrium.

\[
\frac{N_\phi}{R_1} + \frac{N_\theta}{R_2} = P_z. \tag{3}
\]

The problem of determining stresses under unsymmetrical loading reduces to the solution of equations (1), (2), and (3) for given values of the load \( P_\phi \), \( P_\theta \), and \( P_z \).

However, it was stated previously that only axisymmetric loading would be considered in this section. For this type of loading, the stresses are independent of \( \theta \) and \( N_\theta \phi = N_\phi \theta = 0 \). Therefore, the equations of equilibrium reduce to:

\[
\frac{d}{d\phi} (RN_\phi) - R_1 N_\theta \cos \phi = -P_\phi RR_1 \tag{4}
\]

\[
\frac{N_\phi}{R_1} + \frac{N_\theta}{R_2} = P_z. \tag{3}
\]
III STRESS RESULTANTS

By solving equation (3) for $N_\theta$ and substituting the results into equation (4), we obtain a first order differential equation for $N_\phi$ that may be solved by integration. $N_\theta$ can then be obtained by equation (3).

$$N_\phi = \frac{1}{R_2 \sin^2 \phi} \left[ \int R_1 R_2 (P_z \cos \phi - P\phi \sin \phi) \sin \phi \, d\phi + C \right]$$

$$N_\theta = R_2 \left( P_z - \frac{N_\phi}{R_1} \right)$$

The constant of integration "C" represents the effect of loads applied above a parallel circle $\phi = \phi_0$. $2\pi C$ is the resultant of these forces. If the shell is closed, the loading will degenerate to the concentrated radial force $P_z$ at the vertex of the shell. (See Figure B7.1.1.4 - 3a.)

If the shell has an opening, the angle $\phi_0$ defines the opening and the loading (lantern type loading, Figure B7.1.1.4 - 3b) results in the following:

$$N_\phi = -\frac{P}{2\pi R_2 \sin^2 \phi} \quad , \quad N_\theta = \frac{P}{2\pi R_1 \sin^2 \phi}$$

These loads may be treated as additive loads because of the loaded opening at the vertex of the shell. Bending stresses will be introduced at $\phi_0$ but will tend to dissipate rapidly with increasing $\phi$. 
Fig. B7.1.1.4 - 3: Loading above $\phi = \phi$ [Diagram showing closed and open shell loading with $P$, $P_z$, and $\phi$ labels]
### IV STRESS, STRAIN, AND DISPLACEMENT

Once the stress resultants, $N_\phi$ and $N_\theta$, are obtained, stresses, strains, and displacements are readily obtained by the usual methods. For the symmetrically loaded membrane shell, the loading component and displacement in the circumferential direction are zero (Figure B7.1.1.4 - 4).

![Diagram](image)

**Fig. B7.1.1.4 - 4. Loads and Displacements**

- $P_z$ = Radial component of loading acting on differential element
- $P_\phi$ = Component of loading acting in X direction (tangential to meridian)
- $P_\theta = 0$ = Component of loading acting in Y direction (tangential to parallel)
- $w$ = Small displacement of a point in the Z direction (normal to surface)
\( u = \text{Small displacement in X direction (tangential to meridian)} \)

\( v = 0 = \text{Displacement in Y direction (tangential to parallel)} \)

Because of assumptions of membrane theory and axisymmetric loading (e.g., \( t/R \ll 1 \), all moments \( \approx 0 \), and all shearing forces \( \approx 0 \)) the normal stresses can be expressed simply as:

\[
\sigma_\phi = \frac{N_\phi}{t}, \quad \sigma_\theta = \frac{N_\theta}{t}.
\]

The strain components can be found either from \( \sigma_\phi \) and \( \sigma_\theta \) or \( N_\phi \) and \( N_\theta \):

\[
\varepsilon_\phi = \frac{1}{Et} (N_\phi - \mu N_\theta), \quad \varepsilon_\theta = \frac{1}{Et} (N_\theta - \mu N_\phi)
\]

where \( E = \text{Young's modulus} \)

\( t = \text{Thickness of shell} \)

\( \mu = \text{Poisson's ratio} \).

The displacement components are computed next, thereby completing the solution of the shell problem. The general solution for \( u \) is

\[
u = \sin \phi \left[ \int \frac{f(\phi)}{\sin \phi} \, d\phi + C \right]
\]

where \( C \) is a constant of integration to be determined from support conditions and

\[
f(\phi) = R_1 \varepsilon_\phi - R_2 \varepsilon_\theta = \frac{1}{Et} \left[ R_1(N_\phi - \mu N_\theta) - R_2(N_\theta - \mu N_\phi) \right]
\]

\[
= \frac{1}{Et} \left[ N_\phi (R_1 + \mu R_2) - N_\theta (R_2 + \mu R_1) \right].
\]

The displacement \( w \) can then be found from the equation

\[
w = u \cot \phi - R_2 \varepsilon_\theta.
\]
Because the interaction process of two or more shells is often required, displacements are often calculated in terms of $\bar{u}$ and $\bar{w}$, the vertical and horizontal displacements. (See Figure B7.1.1.4 - 5.) Note that when $R_2 = R$, $u = \bar{u}$ and $w = \bar{w}$. The displacement $\bar{u}$ can be found in a manner similar to the solution for $u$. The general solution for $\bar{u}$ is

$$\bar{u} = R \epsilon_\theta \cot \phi - \int \frac{f(\phi)}{\sin \phi} \, d\phi + C$$

where $f(\phi) = R_1 \epsilon_\phi - R_2 \epsilon_\theta$ and $C$ is again a constant of integration determined from support conditions. The horizontal displacement is simply

$$\bar{w} = R \epsilon_\theta$$

\[\text{(u and w known)}\]

$$\bar{w} = w \sin \phi + u \cos \phi$$

$$\bar{u} = -w \cos \phi + u \sin \phi$$

\[\text{Fig. B7.1.1.4 - 5. Displacements}\]
These formulas (Figure B7.1.1.4 - 5) can be used to convert from one form of displacement to the other, depending on the given solution and the requirements of the user.
B7.1.1.4 EQUATIONS

V SUMMARY

Application of the solutions presented in this section can be classified conveniently for the two following cases: general shell of revolution with general axisymmetric load distribution, and general shell of revolution subjected to uniform pressure. Table B7.1.1.4 - 1 presents a summary of solutions for these general cases.

The remainder of this section presents practical shell of revolution problems with various types of axisymmetric loading. Based on the various shell geometries, solutions for \( N_\phi \) and \( N_\theta \) are presented with the force-displacement relationships. The stresses can be calculated directly using the equations in Section B7.1.1.4 - IV.
Table B7.1.1.4 - 1. Summary of Equations, Axisymmetrically Loaded Shells of Revolution, Linear Membrane Theory

![Diagram of shell structure with equations]

General Shell of Revolution
General Axisymmetric Loading

Deflected Position

General Shell of Revolution
Uniform Pressure

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<tbody>
<tr>
<td>( N_\phi = \frac{1}{R_0 \sin \phi} \left[ \int \frac{R_1 R_2 (P_z \cos \phi - P_\phi \sin \phi) \sin \phi , d\phi}{R_1} \right] + C )</td>
<td></td>
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<tr>
<td>( N_{\theta} = R_1 \left( P_z - \frac{N_\phi}{R_1} \right) )</td>
<td></td>
</tr>
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<td>( \sigma_\phi, \sigma_\theta = \frac{N_\phi}{t}, \frac{N_{\theta}}{t} )</td>
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</tr>
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<td>( u = w \cot \phi - \int \frac{R_1 \epsilon_\phi - R_2 \epsilon_\theta}{\sin \phi} , d\phi + C )</td>
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